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Mathematics Bridging Course
Unit 1 – Introduction

1. Introduction

"*A mathematical truth is neither simple nor complicated in itself, it is.*"

´Emile Michel Hyacinthe Lemoine (1840–1912)

We want to accompany you at the beginning of a trip leading to the profound truth in this quote. Mathematics is an amazing, beautiful science in many ways. It is full of an abstract aesthetics, comparable to that which underlies many modern works of art. It is the most abstract art form that we know, and yet mathematics is not an end in itself. Our life as it is today would be unthinkable without mathematics. Space travel, cars, ships, bridges, skyscrapers, cell phones, radio and television, internet – all of that and more much more we would not have without mathematics. Practical, straightforward applicability and intertwined structures of pure beauty; sometimes performing menial tasks, sometimes being highly creative. Doing math can be varied and sweat-inducing; it can produce frustration and wonderful successes. But every craftsperson and every artist has to start by acquiring the basic techniques of the trade and get a feel for it, developing quality and beauty, practicing tirelessly to be able to create masterpieces by oneself.

2. School mathematics vs. university mathematics

"Isn't there only one kind of mathematics?" one might ask. Almost everyone and even most mathematicians would agree. Yet the terms school mathematics and university mathematics do not so much describe different "kinds of mathematics", but different ways of looking at mathematics, or different ways of presenting and learning mathematics. In school – despite recent efforts of focusing on teaching concepts instead of algorithms – a lot of time is spent solving tasks. And that is a good thing, since it allows students to understand the underlying concepts in a learning-by-doing manner. By working from the concrete to the abstract, students in school gain a step-by-step knowledge of concepts and structures in mathematics. At university, things are done differently: Mathematics as a scientific discipline mostly deals with abstract structures. Those are defined by a few basic attributes. Further properties and relations to other structures are derived by proofs, applying strictly logic conclusions to these attributes and to other (already proven) properties. Tasks are usually "only" used to demonstrate or illustrate these abstract structures. One usually works from the abstract to the concrete – exactly the other way around from what happens in school. This causes a lot of students (even those who had an easy time with mathematics in school) to stumble (and sometimes fall) during the first semester of studying mathematics, experiencing what education experts call the "abstraction shock". Those students who study in the mathematics teacher program often have an even harder time with this. Not because they are less able than "pure mathematics" students, but because they often come in with a different expectation. "This is a teacher training program; why are we not learning mathematics the way we need it, to teach mathematics in school?" is a question a lot of university staff has heard. The answer is not an easy one. It boils down to the consensus amongst many educators that teachers should gain a profound knowledge of the field they are about to teach, getting a comprehensive overview of its various areas and methods, so that they will be able to use these funds of knowledge to create rich and motivating lessons for their students in school.

3. Basic building blocks

There is a variety of lists from different mathematicians and other people about the basic building blocks of mathematics. Since this material is neither a philosophical debate nor a lecture about mathematical structures but an attempt to help mathematics teacher students at the beginning of their studies, we will concentrate on two aspects that are known as pitfalls – language and proofs.

3.1 Mathematical language

Let us begin this chapter with a joke from the famous physicist Herbert Pietschmann: A philosopher, a physicist and a mathematician are on vacation in Scotland. They see a sheep that appears to be black.
Philosopher: "*Scottish sheep are black!*"
Physicist: "*Oh no, you cannot say that. You have to say: There are black sheep in Scotland!*"
Mathematician: "*You have it all wrong. You have to say: There is at least one sheep in Scotland which is black on at least one side!*"

Language is there to transmit information, in mathematics as much as in daily life. Yet the language used in mathematics is comparable to its field – it is very structured and extremely precise. While this sounds more like an advantage than a pitfall, many people are not used to this kind of precision in language. Most of us will go with the physicists' answer in the example above, and for most everything in daily life that would be precise enough. For a mathematician however it is important to convey exact information which as little ambiguity as possible. For mathematics students, that can be difficult at the beginning. So it is advisable to practice "translating" symbolic mathematical expressions and statements into full sentences. This is particularly important for mathematics teacher students, since it is an ability they are supposed to convey to their students in school.

Tasks:

Task 1: "Nazife is 20 years older than her daughter Emine."

- a) *Use the variables n for the age of Nazife and e for the age of Emine to express that statement in a mathematical equation!*
- b) *If d is the age of Nazifes son Dušan, express in one sentence what the equation $d = \frac{e}{2} + 1$ means!*

Task 2: "$h \leq k < r$", where h, k and r are the body heights of Hermann, Karim, and Roberto, respectively. Which one(s) of the following statements describe the meaning of this expression in its entirety? Note: Be aware that "Hermann is the shortest of the three" would not be a correct answer, because although it is a true statement, it does not describe the inequality in its entirety.

- a) *Karim is taller than Hermann but shorter than Roberto.*
- b) *Karim is not shorter than Hermann but shorter than Roberto.*
- c) *Roberto is taller than Hermann and Karim.*
- d) *Hermann is shorter than or of equal height as Karim, who is shorter than Roberto.*
- e) *Roberto is taller than Hermann who is shorter than Karim.*
- f) *Roberto is taller than Karim who is taller than Hermann.*

Task 3: Use a suitable function and variables of your choice to describe the (probably very much exaggerated) statement "My income doubles every ten years!"

As any other science, mathematics comes with a variety of technical terms, also called mathematical terminology. Some of these terms students already know from school (e.g. a lot of algebraic and geometrical notations), others they will have to acquire at university. A lot of these terms are specific to a certain area of mathematics and should be taught in lectures concerning this area (it would not make sense to explain to a beginning student what Banach Spaces or Trilinear Coordinates are). But a number of them appear in almost all areas, and it would be prudent to give students a short list of these recurring terms, together with some explanations.

- *Axiom*: A fundamental statement that is accepted as being correct and does not need any further proofing.
- *Theorem*: A statement derived by applying logic conclusions to axioms or other (already proven) theorems. Depending on the perceived importance of a theorem, a number of other words may be used for it:
    - *Fundamental theorem*: A theorem that is seen as being very important for a specific area of mathematics.
    - *Proposition*: A theorem that is seen as being of less importance.
    - *Corollary*: A (often minor) theorem that can be deducted from another (often major) theorem with very little effort.
    - *Lemma*: A theorem that is used in the proof of another (more important) theorem, but has little application outside of this proof. Sometimes also the expressions *auxiliary theorem* or *accessory theorem* are used for this.
- *Proof/Proofing*: The act or result of deriving a true statement by applying logic conclusions to axioms or other already proven theorems.
- *Conjecture*: A statement that is believed to be true but has not yet been proven.
- *Definition*: An exact description of a newly introduced word, term, expression, or symbol, usually done by describing the properties or conditions of the new object.
- *QED*: An acronym for the Latin expression "quod erat demonstrandum", meaning "what was to be shown [or demonstrated]". It is usually written at the end of a proof to signify exactly that: the end of the proof (which is particularly useful for longer mathematical texts with a lot of proofs in them). Also the symbol □ is occasionally used, usually at the end of the last line of a proof.

3.2 Proofs

If you ask a school student why she or he thinks that a certain mathematical statement is correct, the answer is most likely "because the teacher said so", "because it's written in the text book", or "everyone knows that". This means that the need for proving mathematical statements or expressions is a concept that not all beginning students are familiar with. A good way is for students to imagine mathematics as a building that is based on axioms (basic statements that are agreed to be correct and don't need any further proof) as its foundation. Everything else – the walls, ceilings, etc. – are theorems. A theorem is a mathematical statement that has been derived by applying logic conclusions to axioms or other (already proven) theorems. This process is called "proving", its result is called a "proof". So one can only "erect a new wall" (add a new theorem) by building onto the existing structure and using suitable "mortar" (logic conclusions). This image is useful for a variety of reasons. It shows for instance that it is important that all theorems are thoroughly proven, otherwise all other "walls" resting on them could collapse. Also, it is very helpful to know about the "upper floors" (higher mathematics), otherwise one has to start building from the ground every single time one wants to “erect a new wall”. And finally, if one wants to erect the building of mathematics in one’s mind (i.e. learn mathematics), one cannot just start with the third floor and build on thin air, but one has to start with laying the foundations, building the first walls, and gradually proceed to the third floor. In any case, the “building of mathematics” rests on proofs. Students can start with being presented very simple proofs (some of which they may know from school), e.g. of some geometric or algebraic theorems, or theorems from Number Theory. However, as one of our famous mathematics education colleagues, Hans-Christian Reichel, was fond to say, “Mathematics is not a spectator sports!” It is therefore important that students start *doing* proofs by themselves very early on. This can e.g. be done by introducing a theorem with its proof, and then ask students to proof a similar theorem by themselves.

Example 1: “Proof that if $n \in \mathbb{N}$ is an even number, n^2 is also an even number!”

Proof (demonstrated by the lecturer):

If $n \in \mathbb{N}$ is even $\Rightarrow$ there is an $m \in \mathbb{N}$ such that $n = 2m \Rightarrow n^2 = (2m)^2 = 4m^2 = 2\,(2m^2) \Rightarrow$

$\Rightarrow n^2$ is even.

Task for students: “Proof that if $n \in \mathbb{N}$ is an odd number, n^2 is also an odd number!”

Example 2: “Proof that if f is a linear function with $f(x) = k \cdot x + d$, then the differential quotient (i.e. the derivative) at an arbitrary point x is given by $f'(x) = k$!”

Proof (demonstrated by the lecturer):

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{k \cdot z + d - (k \cdot x + d)}{z - x} = \lim_{z \to x} \frac{k \cdot (z - x)}{z - x} = \lim_{z \to x} k = k$$

Task for students: “Proof that if f is a quadratic function with $f(x) = a \cdot x^2 + b \cdot x + c$, then the differential quotient (i.e. the derivative) at an arbitrary point x is given by $f'(x) = 2a \cdot x + b$!”

c) The vast sea of mathematics

The idea of this chapter is to give students a (very!) brief overview of mathematical research areas. This in itself is not an easy task, since different mathematicians may give different answers on how to classify research areas in mathematics. We will therefore go with the leading classification, which is called the Mathematics Subject Classification (version MSC2020), and group the first-level research area classifications there in a frequently used way.

- Mathematical logic and foundations
- Discrete mathematics and algebra
- Analysis
- Geometry and topology
- Applied mathematics

Each of these groups will be covered in a section of this chapter that briefly explains the most important research areas of the groups.

4.1 Mathematical logic and foundations

In this group, the foundations of mathematics are to be found. *Set Theory* discusses mathematical sets (the "lowest level" structure for most mathematicians). *Algebraic Logic* is formalizing logic conclusions and deduction. One meta-level above, *Proof Theory* formalizes proofs as objects rather than actions to allow for an analysis of what counts as a "valid proof" on a syntactic level; correspondingly, *Model Theory* does this analysis on a semantic level. A more recent field in this group is *Computability Theory*, which discusses questions like "what can be computed?" and "what does it mean 'to compute' something?"

4.2 Discrete mathematics and algebra

This group deals mainly with structures in mathematics. We try to give a "typical question" in each of the research fields that students can relate to, or at least name some structures that are used in these fields and that students are familiar with.

Combinatorics looks into the various ways of enumeration and counting ("if in a group of n people everyone shakes hands with everyone else, how many handshakes occur?"). *Graph Theory*, sometimes considered a sub-area of Combinatorics, researches relations between certain pairs of objects that are represented both graphically and as sets of pairs ("how many colors does one need to color any conceivable map, such that two neighboring countries are not colored with the same color?"). *Number Theory* studies integer numbers and functions with integer values. As part of that, also Prime numbers are studied ("how many twin primes are there?").

Several subfields in this group deal with various algebraic systems: *Group Theory* discusses sets with a closed binary operation that fulfills the conditions of associativity, identity, and invertibility (some groups are also commutative; a typical example would be Integers with the operation "addition"). *Ring Theory* (and its subfields) adds a second binary operation (which is at least associative and – together with the first operation – distributive) to a commutative group (a typical example here would be Integers with addition and multiplication); the various subfields look at special ring structures or add additional conditions to the second binary operation. In *Field Theory* invertibility (for every nonzero element) is added to the second binary operation, allowing the operation of division (a typical example here are the Real numbers).

A special field in this group is *Linear Algebra*, a field with which students will be extensively confronted at the beginning of their studies. It deals with linear equations and their solvability, with vectors and their abstraction, and with matrices ("how can one solve the following system of linear equations …?").

4.3 Analysis

This group deals mainly with functions, or – more generally – with formalizations of relations between mathematical objects and descriptions of change. One major field that students are often confronted with at the beginning of their studies is (Real) *Calculus*, i.e. the study of *Real Functions* This includes the study of various properties, e.g. continuity ("does the function graph 'have holes'?"), as well as integrals (one interpretation of which is "what is the area beneath the function graph?") and derivatives ("what is the slope of the function graph?"). *Complex Analysis* (also sometimes called Complex Calculus) studies complex functions in one or more variables.

Measure Theory generalizes the concept of integration by assigning numbers ("measures") to subsets. *Integral Equations* and *Integral Transforms* are further fields studying certain aspects of integration.

Several fields of mathematics study equations where functions and their derivatives appear (the solutions of such equations are hence usually functions, not numbers). *Ordinary Differential Equations* (the word "ordinary" does in no way mean "simple" or "easy" here) contain function(s) of one independent variable and derivatives of this/these function(s). *Partial Differential Equations* can have more than one independent variable. A large variety of real-life problems in various fields of science and other research areas can be described by differential equations.

Several diverse areas in this group have a number of real-life applications. *Dynamical Systems* describe the time dependency of a point (often in 2D or 3D) over time and are used in Biology, Medicine, Economics etc. *Sequences and Series* are often used to describe the development of Dynamical Systems. *Harmonic Analysis* studies functions that can be described as generalizations of waves and are used in several areas of Physics, Electronics, and Neuroscience.

One abstraction level above those functions that are usually looked at in Calculus (which are functions whose objects are [mostly numerical] variables), *Functional Analysis* studies transformations and functions whose objects are functions themselves, thereby combining several areas of mathematics (Calculus, Linear Algebra, Topology and others). *Operator Theory* and *Variation Theory* look into certain varieties of such functional transformations.

4.4 Geometry and Topology

In this group, various abstractions of space are studied. The word *Geometry* is often used today as a superordinate term for various fields of mathematics. It includes *Euclidean Geometry* (where parallel lines never intersect and always have the same distance, or "where everything is as it should be", as a colleague once noted; i.e. the conventional notions and properties of points, planes, angles etc. in our environment apply) and *Non-Euclidean Geometry* (e.g. geometry on the surface of a sphere). *Manyfolds* are generalizations of Euclidean space (they "locally behave like Euclidean space").

Discrete Geometry studies combinatorial properties of discrete (often finite) geometric objects (tessellations and circle packings are for instance studied here, i.e. questions like "with which geometric forms can one completely cover a plain without overlapping?" and "how dense can one cover a plain with circles?"). *Differential Geometry* applies methods of calculus to study geometric objects (e.g. to measure the curvature of surfaces).

Topology studies the behavior of geometric objects under (continuous) deformations. Several subfields, as *Algebraic Topology* or *Differential Topology*, use different fields of mathematics as tools for working with topological spaces.

4.5 Applied Mathematics

Even amongst mathematicians it is debated which fields of mathematics should go into this group (since almost all mathematical research areas have some extra-mathematical applications), or even if this group actually exists. Oftentimes, *Probability Theory* (working with random or stochastic processes) and *Statistics* (data analysis) are found in this group, as are *Numerical Analysis* (studying numeric algorithms, oftentimes approximations), *Mathematical Optimization* (finding a "best" solution) and *Operations Research* (optimizing decisions in complex situations), *Game Theory* (modelling strategic interactions), and *Financial Mathematics*.

Several areas of applied mathematics are applications of mathematics in other fields, with oftentimes very diffuse limits between those fields and mathematics ("is this physics, or is it mathematics?"). Amongst those are *Quantum Theory*, *Relativity*, *Fluid Mechanics*, *Statistical Mechanics*, *Artificial Intelligence*, *Computer Algorithmics* etc.

d) … and finally

For this bridging course, we have several objectives. First, we want to help the students to "bridge the gap" between school mathematics and university mathematics. Second, we want to give a very brief overview of what the various fields of mathematics are all about, so that they can make a choice of lectures where this is possible. And third, we want to introduce them to various aspects of mathematics teaching right at the beginning of their studies. To fulfill all those goals with equal rigor would of course require a bridging course that would fill many days (and nights) for both students and lecturers. We therefore concentrate on the main objective of supporting students to bridge the gap between school and university, and include aspects of the second and third objective into that. For that reason we chose a total of ten mathematical topics, grouped into four modules, which have either proven to be particularly hard for students, or are handled very differently at university as they are in school. Each of the topics will include anchor points in school mathematics, real-life applications if possible, teaching tips, and exercises. We hope that this choice will support students at the beginning of their studies, allowing them to successfully progress, and, finally, becoming motivated mathematics teachers!