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Mathematics Bridging Course
Unit 2a_1 – Logic

PROPOSITIONS, TRUTH VALUES

When observing objects, we can come to some conclusions that we can pronounce or write. This proposition is either true or false, and it makes sense to talk about its truthfulness. Thus, the proposition is any sentence, of which it makes sense to say whether it is true or false. We also say that each statement p is associated with its truth value, which we denote $tv(p)$. If the proposition p is true, the truth value of the proposition is equal to one ($tv(p) = 1$). If the proposition is false, its truth value is zero (we write $tv(p) = 0$). If we are creating a table of truth values of propositions, or we know that we work with truth values of propositions, the word "tv" may be omitted. The requirement that the "proposition is either true or false" is called the theorem of two values and is the foundation of classical binary logic. Propositions expressed by a simple sentence are called elementary propositions. Letters identifying the elementary propositions are called propositional variables.

Operations with propositions

Let p, q be the propositions from the set of propositions V. Then:

a) *Alternative* (resp. *disjunction*) of proposition p, q we read "or", we mark $p \vee q$, is a proposition that is true in all those cases where at least one of the proposition p, q is true.
b) *The conjunction* of the propositions p, q, we read „and", we denote $p \wedge q$, is a proposition that is true only if both propositions p, q are true at the same time.
c) *The implication* of the proposition q by the proposition p we read, "It follows, or if... then..." we mark $p \Rightarrow q$, is a proposition that is false only in the case if p is a true proposition and q is a false proposition. The proposition p in this case we call a prerequisite of implications and proposition q conclusion of the implication.
d) *The equivalence* of propositions p, q, we read „if and only if", we denote $p \Leftrightarrow q$, is a true proposition, if both propositions p, q are true or both propositions p, q are false.
e) *The negation* of the proposition p is a proposition p' that is true if and only if p is a false proposition. We read, "it is not true that p".

Propositions created from propositions p, q using logical connectives (or also the so-called functors), we call compound propositions. Knowledge of the truth values of these composite propositions is expressed in Table 1.

p	q	p'	q'	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
1	1	0	0	1	1	1	1
1	0	0	1	0	1	0	0
0	1	1	0	0	1	1	0
0	0	1	1	0	0	1	1

Table 1: The truth values of compound propositions

Expressions that we construct with the functors $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, $'$ and parentheses in terms of expression building rules are called propositional formulas. We denote $a(p_1, \dots, p_n)$ (or simply a), where $p_1, \dots, p_n$ are the propositional variables.

The proposition arising from the formula, when substituting propositions for all the propositional variables, is called interpretation of the formula. Take, for example, the formula $p \wedge q \Rightarrow r$. Its interpretation for natural numbers can be, for example: "When 2 divides 12 and 3 divides 12, then 6 divides 12.".

The propositional formula is not a proposition, it has no truth value. Each propositional formula, however, we can assign a function that expresses the dependence of truth values of this formula interpretations from the truth values of propositions substituted for individual propositional variables.

The principle of evaluating propositional formulas

If we arrange the basic logical connectives into a sequence $'$, $\wedge, \vee, \Rightarrow, \Leftrightarrow$, then each logical connective standing to the left of the considered right one has a higher priority at the formula evaluation. This principle is also called the principle of priority of logical connectives**.** In the free-standing formulas, the outer parentheses are omitted.

Propositional formula may be from its truth assessment point of view:

a) *A tautology* – it is a propositional formula from which after substituting propositions in propositional variables, we always get a true proposition.
b) *A contradiction* – it is such a propositional formula from which after substituting any propositions in propositional variables we always get a false proposition.
c) *A satisfiable propositional formula* – it is a propositional formula that is neither a contradiction (there exists at least one evaluation 1 in the truth table), thus tautology it is a special case satisfiable propositional formula.

Example 1. We decide using a table of truth values whether the formula $\big(p \wedge (p \Rightarrow q)\big) \Rightarrow q$ is a tautology.

Solution. We create a table of truth values of the given formula (Table 2) and we see the truth-evaluation of the formula $\big(p \wedge (p \Rightarrow q)\big) \Rightarrow q$ is 1 for every evaluation of propositional variables p, q.

p	q	$p \Rightarrow q$	$p \wedge (p \Rightarrow q)$	$\big(p \wedge (p \Rightarrow q)\big) \Rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

Table 2: Truth values of the formula $\big(p \wedge (p \Rightarrow q)\big) \Rightarrow q$

Example 2. John, Mason and Sarah are suspected of stealing. They testified under oath as follows: John: "*Mason is guilty and Sarah is innocent.*" Mason: "*If John is guilty, then Sarah is guilty as well.*" Sarah: "*I am innocent, but at least one of the others is guilty.*" We will show who is guilty.

Solution. We gradually denote by propositional variables j, m, s the propositions "*John is guilty.*", "*Mason is guilty.*", "*Sarah is guilty.*". Then, we write the testimony of the three suspects using propositional formulas $m \wedge s'$, $j \Rightarrow s$, $s' \wedge (j \vee m)$. Using the table of truth values we now determine if these formulas are a satisfiable system (Table 3).

j	m	s	$m \wedge s'$	$j \Rightarrow s$	$s' \wedge (j \vee m)$
0	0	0	0	1	0
0	0	1	0	1	0
0	1	0	**1**	**1**	**1**
0	1	1	0	1	0
1	0	0	0	0	1
1	0	1	0	1	0
1	1	0	1	0	1
1	1	1	0	1	0

Table 3: Truth values of formulas $m \wedge s'$, $j \Rightarrow s$, $s' \wedge (j \vee m)$

All formulas created from testimonies of the suspects are true in the third row of the table. In this row, only the proposition m is truth evaluated as 1, hence, only Mason is guilty.

Example 3. We will show that from the given assumptions:

a) Ivan is an electrician.
b) It is not true that, Ivan is an electrician and he is rich.
c) If Ivan is the owner of a prosperous company, he is rich.

The conclusion follows: Ivan is not the owner of a prosperous company.

Solution. Denote in sequence the propositions:

x: Ivan is an electrician.
y: Ivan is rich.
z: Ivan is the owner of a prosperous company.

Then assumptions: „*Ivan is an electrician.*", „*It is not true that, Ivan is an electrician and he is rich.*", „*If Ivan is the owner of a prosperous company, he is rich.*" can be written as follows: x, $(x \wedge y)'$, $z \Longrightarrow y$.

We create a table of truth values of assumptions (Table 4):

x	y	z	$(x \wedge y)'$	$z \Longrightarrow y$	z'
0	0	0	1	1	1
0	0	1	1	0	0
0	1	0	1	1	1
0	1	1	1	1	0
1	0	0	**1**	**1**	**1**
1	0	1	1	0	0

1	1	0	0	1	1
1	1	1	0	1	0

Table 4: Truth values of formulas x, $(x \wedge y)'$, $z \Longrightarrow y$, z'

In the table (in the fifth row), there exists an evaluation of variables that x, $(x \wedge y)'$, $z \Longrightarrow y$, z' applies, hence, the conclusion is true.

Tautological equivalences

Two formulas a, b with the same variables are called tautologically equivalent if they have the same truth assessment. We denote tautological equivalent formulas $a \sim b$.

Example 4. We show that the formulas $(p \wedge q)'$ and $p' \vee q'$ are tautologically equivalent.

Solution. We create a truth table (Table 5) of both formulas. The formulas have two variables p, q. We see that by the table the identical truth functions are defined, thus the formulas are tautologically equivalent.

p	q	$(p \wedge q)'$	$p' \vee q'$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

Table 5: Truth values of formulas $(p \wedge q)'$, $p' \vee q'$

Let a, b, c be propositional formulas with the same variables. Then, the following tautological equivalences hold (Table 6):

1.	$a \vee b \sim b \vee a$	$a \wedge b \sim b \wedge a$	commutative laws
2.	$(a \vee b) \vee c \sim a \vee (b \vee c)$	$(a \wedge b) \wedge c \sim a \wedge (b \wedge c)$	associative laws
3.	$(a \vee b) \wedge c \sim (a \wedge c) \vee (b \wedge c)$	$(a \wedge b) \vee c \sim (a \vee c) \wedge (b \vee c)$	distributive laws
4.	$(a \vee b)' \sim a' \wedge b'$	$(a \wedge b)' \sim a' \vee b'$	de Morgan's laws
5.	$a \vee a \sim a$	$a \wedge a \sim a$	laws of idempotence
6.	$a \vee a' \sim 1$	$a \wedge a' \sim 0$	laws of complementarity
7.	$(a')' \sim a$	$a \wedge (a \vee b) \sim a$	law of involution
8.	$a \vee (a \wedge b) \sim a$	$a \wedge (a \vee b) \sim a$	laws of absorption
9.	$a \vee 0 \sim a$	$a \wedge 1 \sim a$	laws of identity
10.	$a \vee 1 \sim 1$	$a \wedge 0 \sim 0$	law of unitary alternative and zero conjunction
11.	$a \Rightarrow b \sim b' \Rightarrow a'$	---	law of variation
12.	$a \Rightarrow b \sim a' \vee b$	$(a \Rightarrow b)' \sim a \wedge b'$	---

13.	$a \Leftrightarrow b \sim$ $(a' \vee b) \wedge (a \vee b')$	$a \Leftrightarrow b \sim$ $(a \wedge b) \vee (a' \wedge b')$	---

Table 6: Tautological equivalences of propositional formulas

Example 5. We show that the formulas $x \Leftrightarrow y$ and $(x' \wedge y)' \wedge (x' \vee y)$ are tautologically equivalent.

Solution. We use the truth values table to determine the values of forms $x \Leftrightarrow y$, $(x' \wedge y)' \wedge (x' \vee y)$ for all assessments of variables x, y (Table 7).

x	y	$x \Leftrightarrow y$	x'	$x' \wedge y$	$(x' \wedge y)'$	$x' \vee y$	$(x' \wedge y)' \wedge (x' \vee y)$
1	1	1	0	0	1	1	1
1	0	0	0	0	1	0	0
0	1	0	1	1	0	1	0
0	0	1	1	0	1	1	1

Table 7: Truth values of formulas $x \Leftrightarrow y$, $(x' \wedge y)' \wedge (x' \vee y)$

The table shows that the given formulas are equivalent. Now we will show by equivalent simplifications that it is possible to simplify the second formula to the first one. We compute $(x' \wedge y)' \wedge (x' \vee y) \sim (x \vee y') \wedge (x' \vee y) \sim (y' \vee x) \wedge (x' \vee y) \sim (y \Rightarrow x) \wedge (x \Rightarrow y) \sim x \Leftrightarrow y$.

Example 6. By equivalent simplifications, we show that the formulas $x \Rightarrow (y \Rightarrow x)$ and $(y' \Rightarrow x') \Rightarrow (x \Rightarrow y)$ are tautologies.

Solution. First, we will simplify $x \Rightarrow (y \Rightarrow x)$. It holds $x \Rightarrow (y \Rightarrow x) \sim x \Rightarrow (y' \vee x) \sim x' \vee (y' \vee x) \sim x' \vee y' \vee x \sim x' \vee x \vee y' \sim 1 \vee y' \sim 1$. Now, we simplify $(y' \Rightarrow x') \Rightarrow (x \Rightarrow y)$. It holds $(y' \Rightarrow x') \Rightarrow (x \Rightarrow y) \sim (y \vee x') \Rightarrow (x' \vee y) \sim (y \vee x')' \vee (x' \vee y) \sim (y' \wedge x) \vee (x' \vee y) \sim \big(y' \vee (x' \vee y)\big) \wedge \big(x \vee (x' \vee y)\big) \sim (y' \vee x' \vee y) \wedge (x \vee x' \vee y) \sim (y' \vee y \vee x') \wedge (x \vee x' \vee y) \sim (1 \vee x') \wedge (1 \vee y) \sim 1 \wedge 1 \sim 1$.

Example 7. We will simplify the formula $\big((x \Rightarrow y) \wedge (y \Rightarrow z)\big) \Rightarrow (x \Rightarrow z)$.

Solution. We compute $\big((x \Rightarrow y) \wedge (y \Rightarrow z)\big) \Rightarrow (x \Rightarrow z) \sim \big((x \Rightarrow y) \wedge (y \Rightarrow z)\big)' \vee (x \Rightarrow z) \sim \big((x \Rightarrow y) \wedge (y \Rightarrow z)\big)' \vee (x' \vee z) \sim ((x \Rightarrow y)' \vee (y \Rightarrow z)') \vee (x' \vee z) \sim ((x' \vee y)' \vee (y' \vee z)') \vee (x' \vee z) \sim (x' \vee y)' \vee (y' \vee z)' \vee (x' \vee z) \sim (x \wedge y') \vee (y \wedge z') \vee (x' \vee z) \sim (x \wedge y') \vee \big((y \vee x' \vee z) \wedge (z' \vee x' \vee z)\big) \sim (x \wedge y') \vee \big((y \vee x' \vee z) \wedge (z' \vee z \vee x')\big) \sim (x \wedge y') \vee \big((y \vee x' \vee z) \wedge (1 \vee x')\big) \sim (x \wedge y') \vee \big((y \vee x' \vee z) \wedge 1\big) \sim (x \wedge y') \vee (y \vee x' \vee z) \sim (x \vee y \vee x' \vee z) \vee (y' \vee y \vee x' \vee z) \sim (x \vee x' \vee y \vee z) \vee (y' \vee y \vee x' \vee z) \sim (1 \vee y \vee z) \vee (1 \vee x' \vee z) \sim 1 \vee 1 \sim 1$.

The last two tautological equivalences of the Table 6 allow us to write any propositional formula using only negation, conjunction and disjunction.

Example 8. We will express formulas equivalent to the formula $(x \wedge y) \Rightarrow z'$ so that they contain only: a) the negation and disjunction, b) the negation and conjunction, c) the negation and implication.

Solution. We compute a) $(x \wedge y) \Rightarrow z' \sim (x \wedge y)' \vee z' \sim x' \vee y' \vee z'$, b) $(x \wedge y) \Rightarrow z' \sim x' \vee y' \vee z' \sim (x \wedge y \wedge z)'$, c) $(x \wedge y) \Rightarrow z' \sim (x' \vee y')' \Rightarrow z' \sim (x \Rightarrow y')' \Rightarrow z'$.

Complete disjunctive normal form and complete conjunctive normal form of propositional formulas

Each propositional formula can be expressed:

a) in *disjunctive normal form* as a disjunction of a conjunction of variables, e.g. $a(x, y, z) = (x \wedge y) \vee (x \wedge z') \vee (x \wedge y' \wedge z)$
b) if all variables appear in each term, the formula is in the so-called *complete disjunctive normal form*, e.g. $a(x, y, z) = (x' \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z)$ where the terms of the formula are called *elementary conjunctions*
c) in a *conjunctive normal form* as a conjunction of disjunction variables, e.g. $a(x, y, z) = (x \vee y) \wedge (x \vee y \vee z') \wedge (x' \vee y' \vee z)$
d) if all variables appear in each term, the formula is in the so-called *complete conjunctive normal form*, e.g. $a(x, y, z) = (x \vee y \vee z) \wedge (x' \vee y \vee z') \wedge (x' \vee y' \vee z') \wedge (x' \vee y \vee z)$ where the terms of the formula are called *elementary disjunctions*
e) in a *minimum disjunctive normal form*, which is a disjunctive normal form in which there is a minimum number of terms that contain a minimum number of variables, e.g. $x \vee (y' \wedge z)$
f) in a *minimum conjunctive normal form*, which is a conjunctive normal form in which there is a minimum number of terms that contain a minimum number of variables, e.g. $x \wedge (y \vee z')$

Example 9. We write the formula $r \wedge \big(p' \Rightarrow (r \vee q')\big)$ in the disjunctive normal form (not complete).

Solution. We will use the tautological equivalence $a \Rightarrow b \sim a' \vee b$. It holds that:

$r \wedge \big(p' \Rightarrow (r \vee q')\big) \sim r \wedge \big(p \vee (r \vee q')\big) \sim (r \wedge p) \vee \big(r \wedge (r \vee q')\big) \sim (r \wedge p) \vee (r \wedge r) \vee (r \wedge q') \sim (r \wedge p) \vee r \vee (r \wedge q')$

Example 10. We write the conjunctive normal form (not complete) of formula $(p \wedge q') \Leftrightarrow r$.

Solution. We will use the tautological equivalence $a \Leftrightarrow b \sim (a' \vee b) \wedge (a \vee b')$. It holds that:

$(p \wedge q') \Leftrightarrow r \sim \big((p \wedge q')' \vee r\big) \wedge \big((p \wedge q') \vee r'\big) \sim (p' \vee q \vee r) \wedge (p \vee r') \wedge (q' \vee r')$

Example 11. We determine the complete disjunctive normal form and complete conjunctive normal form of the propositional formula $(x \vee y) \Rightarrow z'$. Further, we minimize them.

Solution. We determine the normal forms using the truth table values of the given formula $f(x, y, z)$ (Table 8).

x	y	z	$x \vee y$	$(x \vee y) \Rightarrow z'$
1	1	1	1	0
1	1	0	1	1
1	0	1	1	0
1	0	0	1	1
0	1	1	1	0
0	1	0	1	1
0	0	1	0	1
0	0	0	0	1

Table 8: The table of truth values of the formula $(x \vee y) \Rightarrow z^{'}$

Using rows of Table 8 where $f(x, y, z) = 1$, we determine the complete disjunctive normal form of the formula. We express the corresponding elementary conjunctions (Table 9).

$(x \vee y) \Rightarrow z'$	elementary conjunctions
0	---
1	$x \wedge y \wedge z'$
0	---
1	$x \wedge y' \wedge z'$
0	---
1	$x' \wedge y \wedge z'$
1	$x' \wedge y' \wedge z$
1	$x' \wedge y' \wedge z'$

Table 9: Elementary conjunctions of the formula $(x \vee y) \Rightarrow z^{'}$

Then, the complete disjunctive normal form of the formula is $(x \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y' \wedge z')$. We minimize the form by equivalent simplifications. We compute $(x \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y' \wedge z') \sim (x \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z') \vee (x' \wedge y' \wedge z) \sim \big((x \wedge z') \wedge (y \vee y')\big) \vee \big((x' \wedge z') \wedge (y \vee y')\big) \vee (x' \wedge y' \wedge z) \sim (x \wedge z') \vee (x' \wedge z') \vee (x' \wedge y' \wedge z) \sim \big(z' \wedge (x \vee x')\big) \vee (x' \wedge y' \wedge z) \sim z' \vee (x' \wedge y' \wedge z) \sim (z' \vee x') \wedge (z' \vee y') \wedge (z' \vee z) \sim (z' \vee x') \wedge (z' \vee y') \wedge 1 \sim z' \vee (x' \wedge y')$.

Now, we determine the complete conjunctive normal form of the formula using the rows of Table 8 where $f(x, y, z) = 0$. We write the corresponding elementary disjunctions (Table 10).

$(x \vee y) \Rightarrow \bar{z}$	elementary disjunctions
0	$x' \vee y' \vee z'$
1	---
0	$x' \vee y \vee z'$
1	---
0	$x \vee y' \vee z'$
1	---
1	---
1	---

Table 10: Elementary disjunction of the formula $(x \vee y) \Rightarrow \bar{z}$

Then, the complete conjunctive normal form of the formula is $(x' \vee y' \vee z') \wedge (x' \vee y \vee z') \wedge (x \vee y' \vee z')$. We get the minimal form by equivalent simplifications. We compute $(x' \vee y' \vee z') \wedge (x' \vee y \vee z') \wedge (x \vee y' \vee z') \sim (x' \vee y' \vee z') \wedge (x' \vee y \vee z') \wedge (x \vee y' \vee z') \wedge (x' \vee y' \vee z') \sim \big((y \wedge y') \vee (x' \vee z')\big) \wedge \big((x \wedge x') \vee (y' \vee z')\big) \sim \big(0 \vee (x' \vee z')\big) \wedge \big(0 \vee (y' \vee z')\big) \sim (x' \vee z') \wedge (y' \vee z')$.

Example 12. We determine the minimal disjunctive form and the minimal conjunctive form of the three variables formula:

a) $f(x, y, z)$, which takes the truth value 0 only for evaluations of (0,0,1), (1,1,0), (1,0,1), (1,0,0)

b) $f(p, q, r)$, which takes the truth value 1 only for evaluations of (1,1,1) and (0,0,1)

Solution. a) All evaluations of given formula are (0,0,0), (0,0,1), (0,1,0), (1,0,0), (1,1,0), (1,0,1), (0,1,1), (1,1,1). Then the formula f takes the value 1 for the evaluations of (0,0,0), (0,1,0), (0,1,1) and (1,1,1). We create a table of elementary disjunctions and elementary conjunctions (Table 11).

x	y	z	$f(x, y, z)$	elementary conjunctions	elementary disjunctions
0	0	0	1	$x' \wedge y' \wedge z'$	---
0	0	1	0	---	$x \vee y \vee z'$
0	1	0	1	$x' \wedge y \wedge z'$	---
0	1	1	1	$x' \wedge y \wedge z$	---
1	0	0	0	---	$x' \vee y \vee z$
1	0	1	0	---	$x' \vee y \vee z'$
1	1	0	0	---	$x' \vee y' \vee z$
1	1	1	1	$x \wedge y \wedge z$	---

Table 11: Elementary conjunctions and disjunction of the formula $f(x, y, z)$

By the disjunction of elementary conjunctions we get the formula $(x' \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y \wedge z) \vee (x \wedge y \wedge z)$ in a complete disjunctive normal form. By the conjunction of elementary conjunctions we get the formula $(x \vee y \vee z') \wedge (x' \vee y \vee z) \wedge (x' \vee y \vee z') \wedge (x' \vee y' \vee z)$ in a complete conjunctive normal form.

By equivalent simplifications, we obtain the minimum forms of the formulas. We compute $(x' \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y \wedge z) \vee (x \wedge y \wedge z) \sim (y' \wedge x' \wedge z') \vee (y \wedge x' \wedge z') \vee (x' \wedge y \wedge z) \vee (x \wedge y \wedge z) \sim \big((y' \vee y) \wedge (x' \wedge z')\big) \vee \big((x' \vee x) \wedge (y \wedge z)\big) \sim \big(1 \wedge (x' \wedge z')\big) \vee \big(1 \wedge (y \wedge z)\big) \sim (x' \wedge z') \vee (y \wedge z)$ and we have the minimum disjunctive form of the corresponding formula of the given function $f(x, y, z)$. Further, we compute $(x \vee y \vee z') \wedge (x' \vee y \vee z) \wedge (x' \vee y \vee z') \wedge (x' \vee y' \vee z) \sim (x \vee y \vee z') \wedge (x' \vee y \vee z') \wedge (x' \vee y \vee z) \wedge (x' \vee y' \vee z) \sim \big((x \wedge x') \vee (y \vee z')\big) \wedge (y \vee x' \vee z) \wedge (y' \vee x' \vee z) \sim \big((x \wedge x') \vee (y \vee z')\big) \wedge \big((y \wedge y') \vee (x' \vee z)\big) \sim \big(0 \vee (y \vee z')\big) \wedge \big(0 \vee (x' \vee z)\big) \sim (y \vee z') \wedge (x' \vee z)$ what already is the minimum conjunctive form of the formula, which corresponds to the given function $f(x, y, z)$.

b) We create a table of elementary disjunctions and elementary conjunctions for formula $f(p,q,r)$ (Table 12).

p	q	r	$f(p,q,r)$	elementary conjunctions	elementary disjunctions
0	0	0	0	---	$p \vee q \vee r$
0	0	1	1	$p' \wedge q' \wedge r$	---
0	1	0	0	---	$p \vee q' \vee r$
1	0	0	0	---	$p' \vee q \vee r$
0	1	1	0	---	$p \vee q' \vee r'$
1	0	1	0	---	$p' \vee q \vee r'$
1	1	0	0	---	$p' \vee q' \vee r$
1	1	1	1	$p \wedge q \wedge r$	---

Table 12: Elementary conjunctions and disjunction of the formula $f(p,q,r)$

By disjunction of elementary conjunctions we get the formula $(p' \wedge q' \wedge r) \vee (p \wedge q \wedge r)$, which can not be further simplified, since the elementary conjunctions do not contain two identical variables. By the conjunction of elementary disjunctions, we get a formula $(p \vee q \vee r) \wedge (p \vee q' \vee r) \wedge (p' \vee q \vee r) \wedge (p \vee q' \vee r') \wedge (p' \vee q \vee r') \wedge (p' \vee q' \vee r)$, which we will further simplify into the minimum form. We compute $(p \vee q \vee r) \wedge (p \vee q' \vee r) \wedge (p' \vee q \vee r) \wedge (p \vee q' \vee r') \wedge (p' \vee q \vee r') \wedge (p' \vee q' \vee r) \sim \big((q \wedge q') \vee (p \vee r)\big) \wedge (p' \vee q \vee r) \wedge (p \vee q' \vee r') \wedge (p' \vee q \vee r') \wedge (p' \vee q' \vee r) \sim \big((q \wedge q') \vee (p \vee r)\big) \wedge \big((q \wedge q') \vee (p' \vee r)\big) \wedge (p \vee q' \vee r') \wedge (p' \vee q \vee r') \sim \big((q \wedge q') \vee (p \vee r)\big) \wedge \big((q \wedge q') \vee (p' \vee r)\big) \wedge (p \vee q' \vee r') \wedge (p' \vee q \vee r') \wedge (p \vee q' \vee r) \sim \big((q \wedge q') \vee (p \vee r)\big) \wedge \big((q \wedge q') \vee (p' \vee r)\big) \wedge \big((r \wedge r') \vee (p \vee q')\big) \wedge (p' \vee q \vee r') \sim \big((q \wedge q') \vee (p \vee r)\big) \wedge \big((q \wedge q') \vee (p' \vee r)\big) \wedge \big((r \wedge r') \vee (p \vee q')\big) \wedge (p' \vee q \vee r') \wedge (p' \vee q \vee r) \sim \big((q \wedge q') \vee (p \vee r)\big) \wedge \big((q \wedge q') \vee (p' \vee r)\big) \wedge \big((r \wedge r') \vee (p \vee q')\big) \wedge \big((r \wedge r') \vee (p' \vee q)\big) \sim \big(0 \vee (p \vee r)\big) \wedge \big(0 \vee (p' \vee r)\big) \wedge \big(0 \vee (p \vee q')\big) \wedge \big(0 \vee (p' \vee q)\big) \sim (p \vee r) \wedge (p' \vee r) \wedge (p \vee q') \wedge (p' \vee q) \sim \big((p \wedge p') \vee r\big) \wedge (p \vee q') \wedge (p' \vee q) \sim r \wedge (p \vee q') \wedge (p' \vee q)$.

Propositional functions

The sentence that contains one or more variables (i.e., it is not a proposition), from which, after substituting permissible values for variables a proposition is created, is called propositional function. We denote it $A(x_1, \dots, x_k)$, where $x_1, \cdots, x_k$ are variables. For example, the sentence "*The number x is an odd number.*" is not a proposition. If we use a specific integer for x, we get the proposition.

The set of all those k entities of elements that are permissible for a propositional function, is called the domain of definition of the propositional function. The set of all k entities of elements of the propositional function domain of definition, for which the proposition is true, is called the truth set (or domain) of the propositional function. For example, $x^2 + y^2 \leq 0$ is a propositional function of two variables x, y. Its domain of definition is $\mathbb{R}^2$. Only the pair $(0,0)$ belongs into the truth domain, hence $\{(x,y) \in \mathbb{R}^2;\ x^2 + y^2 \leq 0\} = \{(0,0)\}$.

A proposition may be created from the propositional function in a different way than substituting values for variables. We can create a proposition, in which is determined the number of elements that when substituted for the variables in propositional function, we get a true proposition. Such propositions are called quantified propositions.

Let $A(x)$ be the propositional function of one variable x with the domain of definition D. Then we can create such quantified propositions:

There exists (There exists at least one) x such that $A(x)$.
There exist at least "n" x such that $A(x)$.
There exist at most "n" x such that $A(x)$.
There exists one and only one "n" x such that $A(x)$.
For all x (it holds) $A(x)$.

The expressions "there exists" and "for all" are called quantifiers. The first one is called existential and second universal. The existential quantifier is denoted by a symbol $\exists$. The quantified proposition "There exists x such that $A(x)$." is denoted $\exists x \in D\ A(x)$. If the domain of definition of variable x is known, also expression $\exists x\ A(x)$ can be used. Especially the proposition "There exists one and only one x such that $A(x)$." is represented by the symbol $\exists!$ The universal quantifier is denoted by the symbol $\forall$. The quantified proposition "For all $A(x)$." is denoted as $\forall x \in D\ A(x)$. If the domain of definition of variable x is known, the expression $\forall x\ A(x)$ can be also used.

Example 13. We decide on the truth of the propositions:

$$\text{a) } \exists n \in \mathbb{Z}\ n^2 = 9,\ \text{b) } \forall n \in \mathbb{Z}\ n^2 = 9,\ \text{c) } \exists n \in \mathbb{Z}\ n^2 = 2,$$
$$\text{d) } \forall n \in \mathbb{Z}\ n^2 = 2$$

Solution. a) The proposition is true, since for example for $n = -3$ it holds $(-3)^2 = 9$. b) The proposition is false, since the equality $n^2 = 9$ does not hold for every integer n, e.g. for $n = 0$ it holds $n^2 = 0^2 = 0 \neq 9$. c), d) The propositions are false since only for two real numbers applies that their second power is 2. Those are numbers $\sqrt{2}$ and $-\sqrt{2}$ but neither one of them is an integer.

We can also create quantified propositions from the propositional functions of two, three or more variables. E.g. the notation $\exists x \in \mathbb{Z}\ \forall y \in \mathbb{R}\ x + y^2 \leq -4$ is a proposition that we read "*There exists an integer x such that for all real numbers y it holds $x + y^2 \leq -4$.*". The proposition $\forall y \in \mathbb{R}\ \exists x \in \mathbb{Z}\ \forall z \in \mathbb{R}\ x + y^2 - z \leq -4$ we read "*For all real numbers y there exists an integer x such that for all real numbers z it holds $x + y^2 - z \leq -4$.*".

By changing the order of the existential and universal quantifier we receive different propositions that may not be equivalent. We can change only the sequence of quantifiers of the same type (both universal or both existential) and the proposition will not change. Therefore, we reduce the expressions $\forall x \in D\ \forall y \in D$ or $\exists x \in D\ \exists y \in D$ just to $\forall x, y \in D$ or $\exists x, y \in D$, respectively.

The principle of quantified propositions negating is presented in Table 13.

Proposition	**Proposition negation**
$\exists x \in D\ A(x)$	$\forall x \in D\ A'(x)$
There exist at least k elements x such that $A(x)$.	There exist at most $k-1$ elements x such that $A(x)$.
There exist at most k elements x such that $A(x)$.	There exist at least $k+1$ elements x such that $A(x)$.
There exist exactly k elements x such that $A(x)$.	There exist more than $k-1$ or at least $k+1$ elements x such that $A(x)$.
$\forall x \in D\ A(x)$	$\exists x \in D\ A(x)$

Table 13: Negation of quantified propositions

We can get the negation of a proposition, which contains some existential and universal quantifiers so that we turn every existential quantifier into a universal one and the universal quantifiers into existential ones and negate the propositional function.

Example 14. We create the negations of propositions:

a) $\exists y \in \mathbb{R}\ \forall x \in \mathbb{R}\ xy \leq y$
b) $\forall x \in \mathbb{N}^+\ \exists y \in \mathbb{N}^+\ \exists z \in \mathbb{N}^+\ x^2 + y^2 = z^2$

Solution. a) $\forall y \in \mathbb{R}\ \exists x \in \mathbb{R}\ xy > y$, b) $\exists x \in \mathbb{N}^+\ \forall y \in \mathbb{N}^+\ \forall z \in \mathbb{N}^+\ x^2 + y^2 \neq z^2$

Example 15. We negate the following propositions (which are either true or false):

a) Number 3 is a root of the equation $x^2 = 9$.
b) $2^3 - 5 > 13$.
c) Diagonals of the square are perpendicular to each other.
d) It holds $-3.5 \in \mathbb{N}$.
e) Each of us speaks Chinese.
f) There is at least one rectangle that has perpendicular diagonals.
g) This tree has at most 20 apples.
h) Each performance has its end.
i) At the meeting, there were just 10 members.
j) $\exists x \in \mathbb{Z}: x^2 - 16 = 0$.
k) $\forall n \in \mathbb{N}: n > 0$.

Solution.

a) Number 3 is not a root of the equation $x^2 = 9$.
b) $2^3 - 5 \leq 13$.
c) Diagonals of the square are not perpendicular to each other.
d) It holds $-3.5 \notin \mathbb{N}$.
e) At least one of us does not speak Chinese.
f) All rectangles do not have perpendicular diagonals.
g) This tree has at least 21 apples.
h) There is at least one performance that has not its end.

i) At the meeting, there were at most 9 or at least 11 members.
j) $\forall x \in \mathbb{Z}: x^2 - 16 \neq 0$.
k) $\exists n \in \mathbb{N}: n \leq 0$.