Co-funded by the
Erasmus+ Programme
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Mathematics Bridging Course
Unit 2a_2 – Sets

Sets and Venn diagrams

The set is one of the most commonly used terms in present mathematics. Foundations of set theory, in the form in which it is still traditional in non-axiomatized form today, were laid and built upon during 1873 – 1884 by German mathematician Georg Cantor (1845 - 1918). Despite initial criticism by Cantor's opponents, his theory became a recognized part of mathematics at the beginning of the 20^{th} century.

If an object m is an element of set M, we denote this as $m \in M$. If an object m is not an element of set M, we denote this as $m \notin M$. We say that set M is well-defined if we can decide unambiguously with respect to each object whether it is an element of set M or not. An empty set is a set that contains no elements. We denote it as $\emptyset$. We call set A a subset of set B if and only if each element of set A is also an element of set B. We denote this as $A \subset B$. We say that sets A, B are equal if and only if it holds that $A \subset B$, $B \subset A$. We denote this as $A = B$. If, at the same time, it does not hold that $A \subset B$, $B \subset A$, then set A, B are not equal. We denote this as $A \neq B$.

Relations between sets can be graphically depicted using the so called Venn diagram conceived by British philosopher John Venn (1834 - 1923). He introduced this diagram for visual representation of sets, their elements and mutual logical relations in 1880 and it consists of circular or elliptical areas that represent a group of objects with a certain common characteristic. Similar illustrations were used in logic even before Venn, e.g. by mathematicians Gottfried Leibniz or Leonhard Euler. However, Venn was the first one who studied them systematically and generalized their use. In practice, Venn diagrams are used for five sets at the most. In 2001, mathematician Peter Hamburger and artist Edit Hepp compiled a diagram for as many as 11 sets. Venn diagrams illustrate different set situations and are a suitable means for solving set problems.

Let us now consider a non-empty set M and its subsets A, B, C, D. The fact that set A is a subset of set M can be graphically depicted as follows (Figure 1):

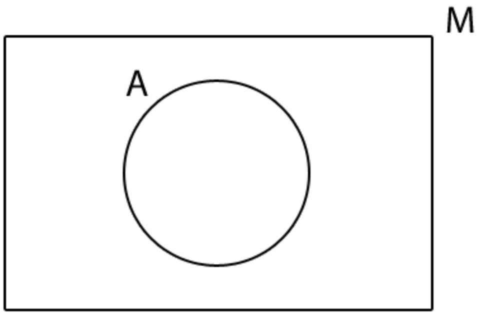

Figure 1: $A \subset M$

The fact that two sets A, B are subsets of set M, would be depicted as follows (Figure 2):

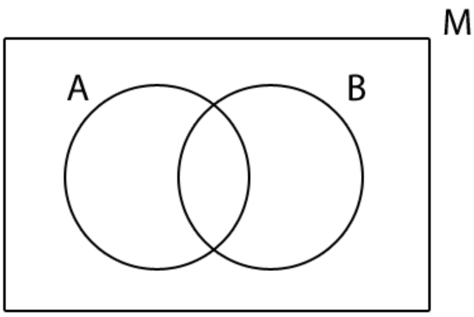

Figure 2: $A \subset M, B \subset M$

Venn diagram for three subsets A, B, C would look like this (Figure 3):

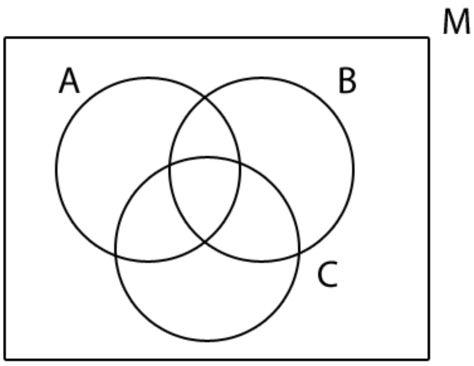

Figure 3: $A \subset M, B \subset M, C \subset M$

Venn diagram for four subsets A, B, C, D would look like this (Figure 4):

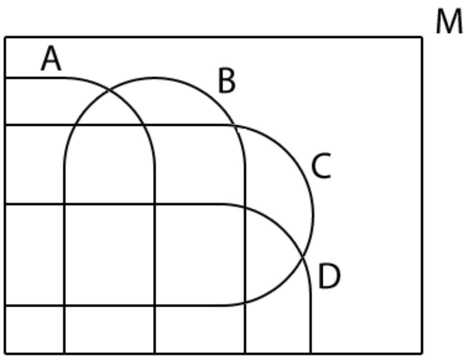

Figure 4: $A \subset M, B \subset M, C \subset M, D \subset M$

A rectangle, which is a representation of set M, divided by subsets to areas, is called the universe of Venn diagram. The universe is also one of the areas.

For every two sets A, B, there is a well-defined set which is called the union of sets A, B (Figure 5) and is denoted as $A \cup B$. This is a set of all those elements of set M which belong to set A or to set B. “Or” is understood in a non-exclusionary sense here, i.e. the union of sets $A \cup B$ includes also elements that belong to both sets A and B).

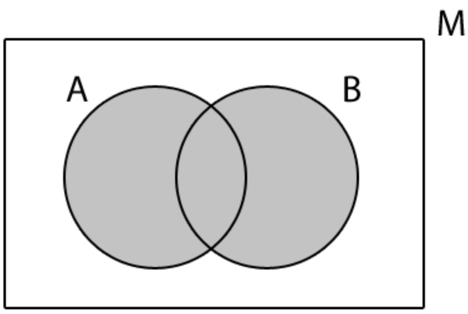

Figure 5: $A \cup B$

For every two sets A, B, there is a well-defined set which is called the intersection of sets A, B and is denoted as $A \cap B$. This is a set of all those elements of set M which are both elements of set A and elements of set B (Figure 6).

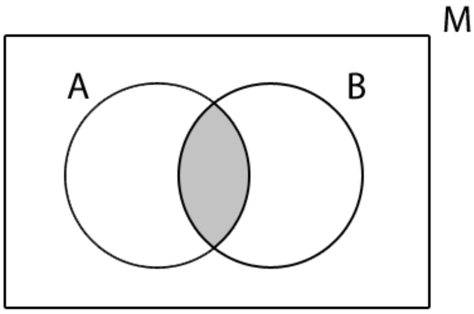

Figure 6: $A \cap B$

For each set A, there is a well-defined set A' which is called the complement of set A with respect to set M. This is a set of all those elements of set M not in A, in other words those, for which it holds that $A \cap A' = \emptyset$, $A \cup A' = M$ (Figure 7).

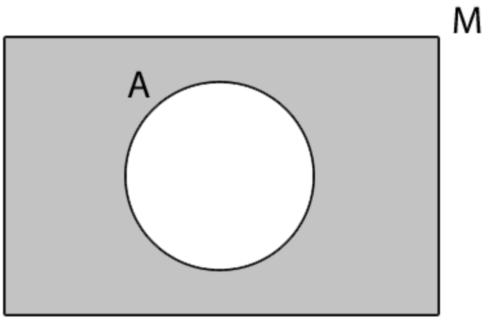

Figure 7: A'

For all subsets A, B, C of set M it holds that $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$. Therefore, it follows that when creating a union and intersection of three sets, parenthesizing does not matter. Therefore, it holds that $(A \cup B) \cup C = A \cup (B \cup C) = A \cup B \cup C$, as well as $(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C$.

Example 1. Let P_1 be the set of all real roots of equation $x^2 + 5x + 6 = 0$, and P_2 be the set of all real roots of equation $x^2 + 4x + 3 = 0$. We want to determine $P_1 \cap P_2$, $P_1 \cup P_2$, P_1'.

Solution. According to the definition of the intersection of sets, $x \in P_1 \cap P_2$ holds if and only if $x \in P_1$ and $x \in P_2$, or if and only if $x^2 + 5x + 6 = 0$, $x^2 + 4x + 3 = 0$ also holds. Set $P_1 \cap P_2$ is then equal to the set of all real roots of the system of equations:

$$x^2 + 5x + 6 = 0$$
$$x^2 + 4x + 3 = 0$$

We will solve this and find that it is a one-element set $\{-3\}$, or $P_1 \cap P_2 = \{-3\}$. Set $P_1 \cup P_2$ equals to the set of all real roots of equation:

$$(x^2 + 5x + 6) \cdot (x^2 + 4x + 3) = 0$$

Therefore it is set $P_1 \cup P_2 = \{-1, -2, -3\}$. And set P_1' is the set of all real numbers x for which it holds $x^2 + 5x + 6 \neq 0$.

Example 2. We want to simplify the notation of set $[(A \cup B) \cap (A \cap B')] \cup (A \cap B)$ so that it would contain as few symbols as possible.

Solution. Using Venn diagram for sets A, B. First, we will mark the area that is a representation of set $(A \cup B)$ (Figure 8).

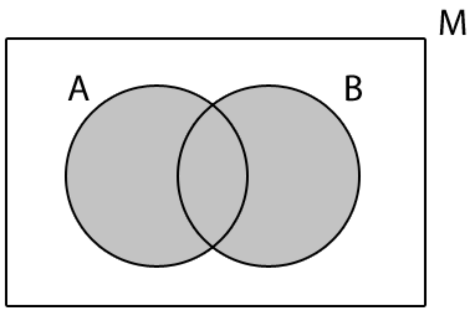

Figure 8: $A \cup B$

Then we will mark the area representing $(A \cap B')$ (Figure 9).

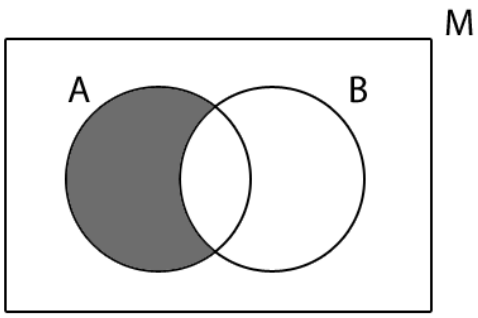

Figure 9: $(A \cap B')$

And then a representation of set $A \cap B$ (Figure 10).

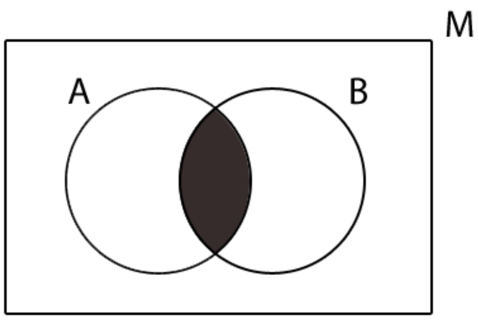

Figure 10: $A \cap B$

Now, we will create a representation of set $[(A \cup B) \cap (A \cap B')]$ which consists of all those areas of Venn diagram that are also in Figure 8 and Figure 9 (Figure 11).

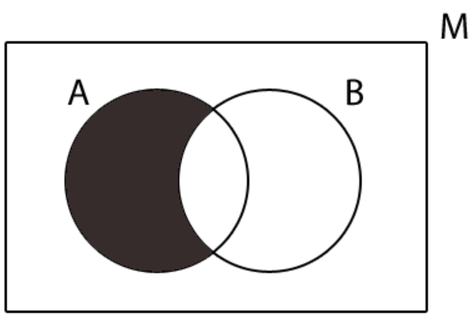

Figure 11: $[(A \cup B) \cap (A \cap B')]$

Then the representation of the set in question $[(A \cup B) \cap (A \cap B')] \cup (A \cap B)$ is the area consisting of all those areas of the Venn diagram marked in Figure 11 or of all the areas marked in Figure 10. We can see that the investigated set is the whole set A, therefore it holds $[(A \cup B) \cap (A \cap B')] \cup (A \cap B) = A$ (Figure 12).

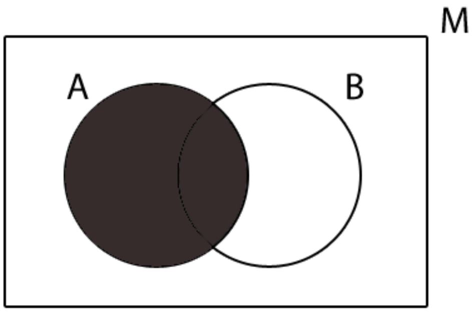

Figure 12: $[(A \cup B) \cap (A \cap B')] \cup (A \cap B) = A$

Example 3. We want to consider set M of all quadrilaterals in the plane. By the letter A we will denote a set of all equilateral quadrilaterals, by the letter B we will denote a set of all quadrilaterals whose all internal angles are right angles. We want to decide whether it holds:

a) $A \cap B$ is the set of all quadrilaterals
b) $A \cup B$ is the set of all quadrilaterals that have at least two axes of symmetry
c) B' is the set of all quadrilaterals whose at least one internal angle is acute or obtuse

Then we will determine sets d) $A \cap B'$ and e) $A' \cup B'$.

Solution. Since set M is the set of all quadrilaterals in the plane, it holds that $M = \{squares, rhombuses, rectangles, rhomboids, kites, irregular\ quadrilaterals, \cdots\}$. Furthermore, A is the set of all equilateral quadrilaters, in other words $A = \{squares, rhombuses\}$, $B = \{squares, rectangles\}$. Furthermore, B is the set of all quadrilaterals with all right internal angles. Now we will create a Venn diagram for sets A, B, M (Figure 13).

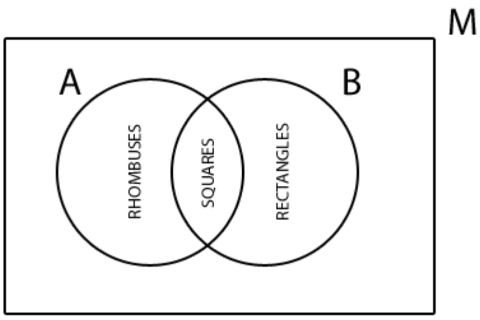

Figure 13: Vennov diagram pre množiny A, B, M

a) Figure 14 shows set $A \cap B$, and therefore it holds that $A \cap B$ is the set of all squares.

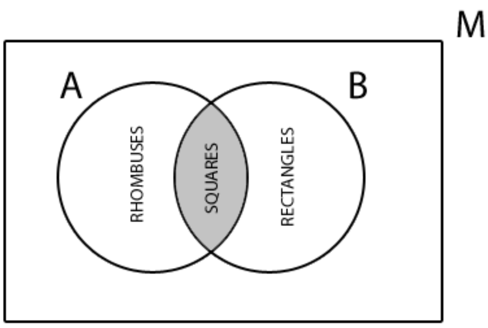

Figure 14: $A \cap B$

b) Figure 15 shows set $A \cup B$, and therefore it holds that $A \cup B$ is the set of all quadrilaterals that have at least two axes of symmetry.

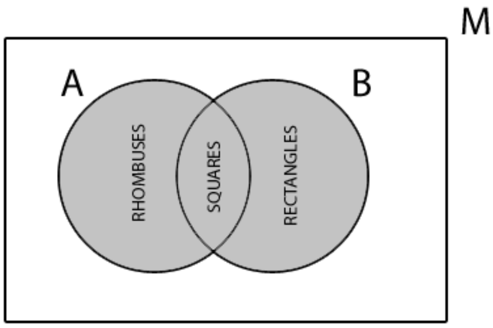

Figure 15: $A \cup B$

c) Figure 16 shows set B' and it holds that B' is the set of all quadrilaterals whose at least one internal angle is acute or obtuse.

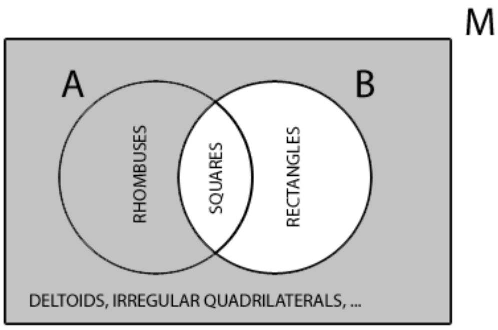

Figure 16: B'

d) Figure 17 shows the area of set $A \cap B'$. So this is the set of all equilateral quadrilaterals whose at least one internal angle is acute or obtuse (the set of all rhombuses).

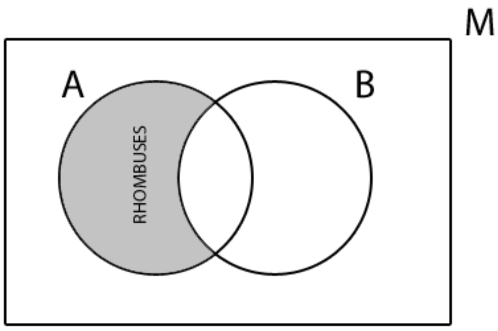

Figure 17: $A \cap B'$

e) Figure 18 shows the areas of set $A' \cup B'$ that represent the set of all quadrilaterals that are not equilateral or whose at least one internal angle is acute or obtuse (so it is the set of all quadrilaterals which are not squares).

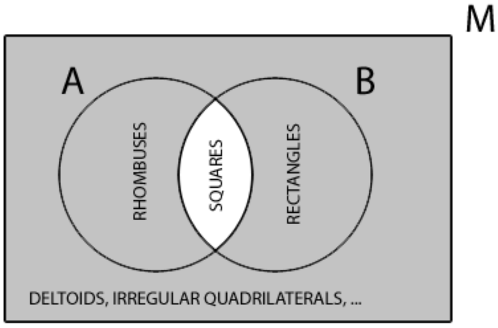

Figure 18: $A' \cup B'$

Example 4. Let A, B, C be subsets of set M. We want to determine a necessary and sufficient conditon so that it holds that $[(A \cup B) \cap (C \cup A)'] \cup (A \cap C') = (C' \cap B) \cup A$.

Solution. We will gradually display the individual sets using Venn diagrams. First we will create representations of $(A \cup B)$ and $(C \cup A)$ and complement of $(C \cup A)'$ (Figure 19).

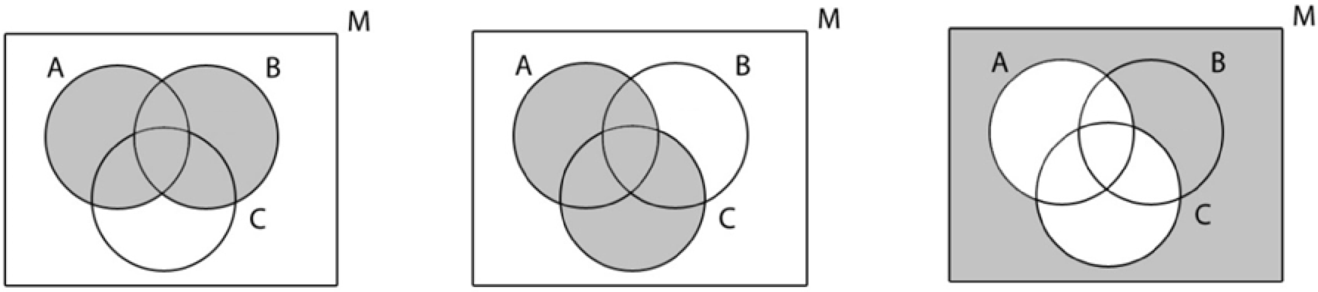

Figure 19: Sequentially $(A \cup B)$, $(C \cup A)$, $(C \cup A)'$

Then the by the intersection of areas of the first and the third diagram, or the areas of set $[(A \cup B) \cap (C \cup A)']$ are shown in Figure 20.

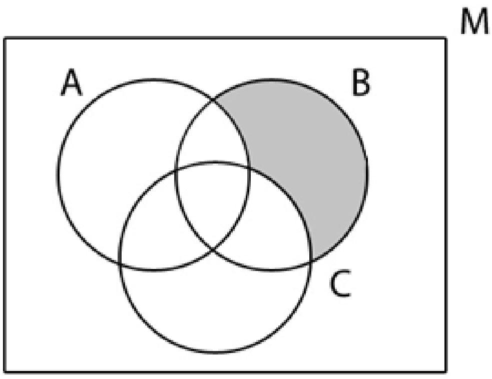

Figure 20: $[(A \cup B) \cap (C \cup A)']$

Now, we will mark $(A \cap C')$ in Figure 21.

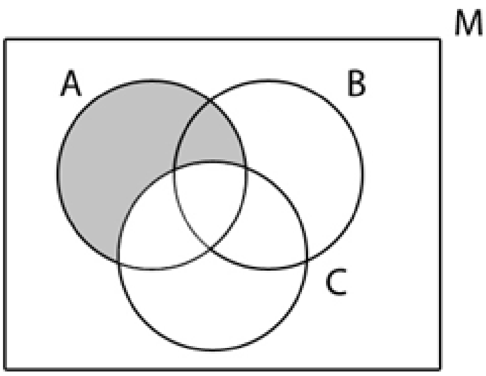

Figure 21: $(A \cap C')$

Then the set defined by expression $[(A \cup B) \cap (C \cup A)'] \cup (A \cap C')$ on the left-hand side is created by the union of areas indicated in Figure 20 and Figure 21 (Figure 22).

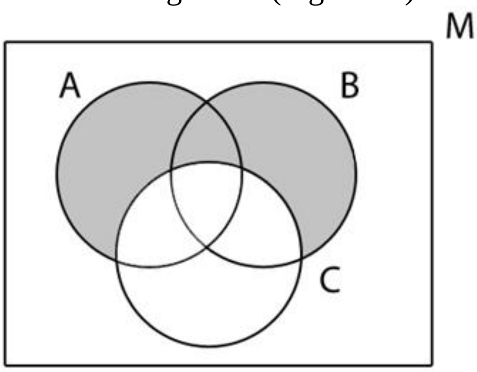

Figure 22: $[(A \cup B) \cap (C \cup A)'] \cup (A \cap C')$

Then we will display the areas of set $(C' \cap B)$ which is included within the right-hand side part of the expression (Figure 23).

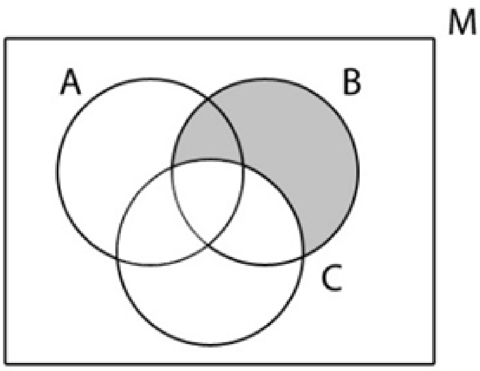

Figure 23: $(C' \cap B)$

Then the diagram of set $(C' \cap B) \cup A$ will look like this (Figure 24):

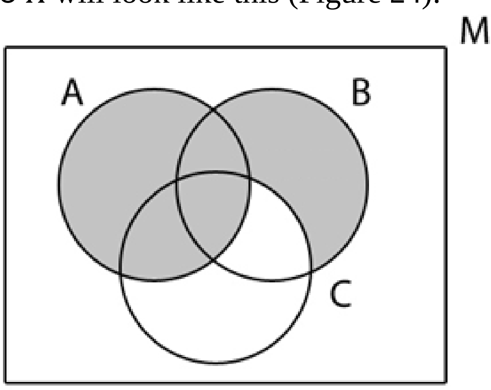

Figure 24: $(C' \cap B) \cup A$

For the sets $[(A \cup B) \cap (C \cup A)'] \cup (A \cap C')$, $(C' \cap B) \cup A$ to be equal, they must consist of the same elements. If we compare the diagrams in Figure 22 (areas of set $[(A \cup B) \cap (C \cup A)'] \cup (A \cap C')$) and Figure 24 (areas of set $(C' \cap B) \cup A$), we can see that the equality of the sets occurs if and only if $A \cap C = \emptyset$ or in other words when the intersection of sets A and C contains no elements, which is a necessary and sufficient condition for the validity of $[(A \cup B) \cap (C \cup A)'] \cup (A \cap C') = (C' \cap B) \cup A$.