Co-funded by the Erasmus+ Programme of the European Union

Mathematics Bridging Course
Unit 2b – Geometry

1.1. Geometric Reasoning

The focus of Geometry and its application continues to evolve with time. Today all teacher and researchers emphasise the important to study Geometry linking it to visualization competencies, problem-solving, problem posing activities, inductive and deductive reasoning, in everyday realities. The dynamic nature of the geometric construction process means that many possibilities can be considered, thereby encouraging exploration of a given problem or the formulation of conjectures. Geometry itself hasn't changed: the introduction in recent years of inexpensive dynamic geometry software programs has added visualization and individual exploration to the study of geometry.

This chapter discusses some geometric concepts/ideas showing how they are part of everyone's life t be experienced in the classroom, at home, or in workplaces.

According to the theory of van Hiele (1999), there are five levels of understanding spatial concepts through which children move sequentially on their way to geometric thinking: visualization, analysis, abstraction, deduction and rigor.

Level 1 (Visualization): students recognize figures by appearance alone; often by comparing them to a known prototype, the properties of a figure are not perceived; students make decisions based on perception, not reasoning.

Level 2 (Analysis): students see figures as collections of properties; can recognize and name properties of geometric figures, but they do not see relationships between these properties; when describing an object, a students might list all the properties they knows, but not discern which properties are necessary and which are sufficient to describe the object.

Level 3 (Abstraction): students perceive relationships between properties and between figures; can create meaningful definitions and give informal arguments to justify their reasoning; logical implications and class inclusions, such as squares being a type of rectangle, are understood; the role and significance of formal deduction, however, is not understood.

Level 4 (Deduction): students can construct proofs; understand the role of axioms and definitions, and know the meaning of necessary and sufficient conditions; students should be able to construct proofs such as those typically found in a high school geometry class.

Level 5 (Rigor): students understand the formal aspects of deduction, such as establishing and comparing mathematical systems; can understand the use of indirect proof and proof by contrapositive, and can understand non-Euclidean systems (cited in Mason, 1999).

This chapter, focusing, in not a exhaustive way, to “explore” some important concept related to a variety of geometric topics, is aimed to define a useful starting point for discussing with students teacher and help them to think about teaching geometry at School. The discussed concepts are refereed to Euclidean Geometry and Transformations.

As the picture suggests, Geometry is in fact around us, as teachers we have to “use” our realty to better live many geometric concept. “ *What are the different geometrical shapes in the picture*?”, “*In the first symbol, what is the ratio between black and white area?*”

According to this prospective, in many part of this document we are indicating possible application of the discussed content in real life.

he first symbol, what is the ratio between black and white areas?”.

In some case we refer to Geogebra as one of the most appropriate and available software for school geometry education in recent years[1].

1.2. Euclidian Elements … axioms

The *Elements* of Euclid were written around 300 BC. It is a collection of thirteen books. Of these, the first six may be categorized as dealing respectively with triangles, rectangles, circles, polygons, proportion and similarity. The next four deal with the theory of numbers. Book XI is an introduction to solid geometry, while XII deals with pyramids, cones and cylinders. The last book is concerned with the five regular solids. Book I begins with twenty three definitions in which Euclid attempts to define the notion of *point*, *line*, *circle* etc. Then the fundamental idea is that all subsequent theorems – or Propositions as Euclid calls them – should be deduced logically from an initial set of assumptions. In all, Euclid proves 465 such propositions in the Elements. These are listed in detail in many texts and in several web-sites devoted to them.

https://mathcs.clarku.edu/~djoyce/java/elements/elements.html is a very interesting attempt at putting Euclid’s *Elements* on-line using some very clever Java applets to allow real time manipulation of figures.

In the *Elements, a*ny initial set of assumptions should be as self-evident as possible and as few as possible so that if one accepts them, then one can believe everything that follows logically from them.

[1] Cfr. http://www.itslearning.net/geogebra

Euclid introduces two kinds of assumptions:

Common Notion:

- Things which are equal to the same thing are also equal to one another.

- If equals be added to equals, the wholes are equal.

- If equals be subtracted from equals, the remainders are equal.

- Things which coincide with one another are equal to one another.

- The whole is greater than the part.

Postulates*:* let the following be postulated.

- To draw a straight line from any point to any point.

- To produce a finite straight line continuously in a straight line.

- To describe a circle with any center and distance.

- That all right angles are equal to one another.

- That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines if produced indefinitely, meet on that side on which are the angles less than two right angles.

Today we usually refer to all such assumptions as *axioms*.

The first four of these postulates too seem self-evident; one surely needs these constructions and the notion of perpendicularity in plane geometry. The Fifth postulate is of a more technical nature, however. To understand what it is saying we have to recall need the notion of parallel lines.

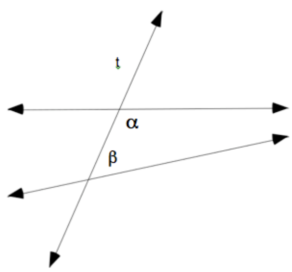
Hilbert many years after Euclid refined axioms 1 and 5 as follows:

1. *For any two different points, (a) there exists a line containing these two points, and (b) this line is unique.*

5. For any line L and point p not on L, (a) there exists a line through p not meeting L, and (b) this line is unique.

Definition. *Two straight lines in a plane are said to be parallel if they do not intersect, i.e., do not meet.*

The Fifth postulate, therefore, means that straight lines in the plane are not parallel when there is a transversal *t* such that the sum $(\alpha + \beta)$ of the interior angles on one side is less than the sum of two right



angles; in fact, the postulate states that the lines must meet on this side.

The figure above makes this clear. The need to assume this property, rather than showing that it is a consequence of more basic assumptions, was controversial even in Euclid's time. He himself evidently felt reluctant to use the Fifth postulate, since it is not used in any of the proofs of the first twenty-eight propositions in Book I. Thus one basic question from the time of Euclid was to decide if the Fifth Postulate is independent of the Common Notions and the first four Postulates or whether it could be deduced from them. Attempts to deduce the Fifth postulate from the Common Notions and other postulates led to many statements logically equivalent to it. One of the best known is **Playfair's Axiom**: *Through a given point, not on a given line, exactly one line can be drawn parallel to the given line.*

In the 19th century, Carl Friedrich Gauss, János Bolyai, and Nikolay Lobachevsky all began to experiment with this postulate arriving at new, non-Euclidean, geometries.

1.3. The axioms of continuity

Two more important axioms that allow us to establish a one-to-one correspondence between the points of a line and the real numbers that preserves the ordering.

R1. (Archimedes) If A and B are two points of a ray |OX, then there is a finite set of points {A1,A2,...,Ak} on |OX such that

$A \in |OA_1|, A_1 \in |OA_2|, \ldots, A_{k-1} \in |OA_k|$, $|OA| \equiv |AA_1| \equiv |A_1A_2| \equiv \cdots \equiv |A_{k-1}A_k|$, *and* $B \in |OA_k|$.

R2. (Cantor-Dedekind) Given a line and two sequences of points $A_1,A_2,A_3,\ldots$, $B_1,B_2,B_3,\ldots$ *on this line such that for every j the segment* $|A_{j+1}B_{j+1}|$ *is contained in the segment* $|A_j B_j|$*, then there exists a point P contained in all of these segments.*

1.4. Euclid's fifth postulate

Parallel lines

E1. (Euclid's fifth postulate) *Given a line l and a point A that does not belong to l, there is a unique line l' passing through A such that l and l' are parallel: l||l' .*

Theorem. **Given two lines l_1 and l_2 that are intersected by a third line l as shown in Figure 2.1, then l||l' if and only if ∠α ≡ ∠β, where ∠α and ∠β are a pair of alternate interior angles.**

Proof. ⇒We startwith l1||l2. Constructaline l3 throughtheintersection of l2 and lsuchthat l3 and l1 form congruent alternate interior angles with l. Then l3 and l1 areparallel, by Proposition 1.3.1, so by Euclid's fifth postulate, l3 = l2. Hence ∠α ≡ ∠β.

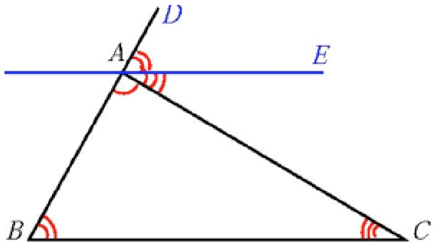
Corollary. Two pairs of parallel lines form congruent angles.

Theorem. The sum of the angles of a triangles is congruent to a straight angle.

Proof. Let ΔABC be a triangle. On AB choose a point D such that A is between B and D. Take the only line through A that is parallel to BC, as shown in Figure 2.2. This line divides ∠CAD into two angles. Let these angles be ∠CAE and ∠EAD. Then by Theorem 2.1.1,

∠CAE ≡ ∠ACB and ∠EAD ≡ ∠ABC.

The conclusion follows, since the angles ∠BAC, ∠CAE and ∠EAD add up to a straight angle.

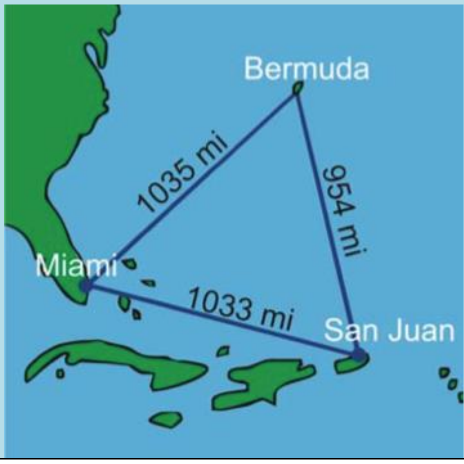


1.5. Triangle, similarity and congruence

Problem 1.1

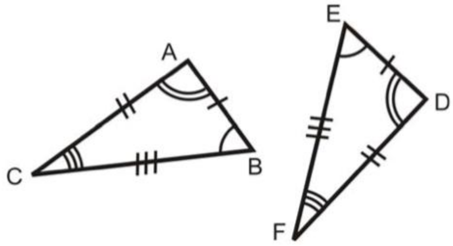
Below is the Bermuda Triangle. You are probably familiar with the myth of this triangle: several ships and planes passed through and mysteriously disappeared.

If you remember, triangle can be classified by its sides or angles. *What type of triangle is this? Classify it by its sides and angles.*



The application this subject to real word are many. A possible starting point to discuss this point with students should be, for example, to show in classroom a video, as the linked below: https://www.youtube.com/watch?v=aejEwpxfKu8, https://www.youtube.com/watch?v=a6MKwW4xO6I

Congruent Triangles: Two triangles are congruent if the three corresponding sets of angles and sides are congruent. "Congruence" is the notion of equality in Euclidean geometry, in the same way as "isomorphic" is the notion of equality in group theory. Congruence is denoted by ≡.



The following congruence theorems specify the conditions under which triangles could be congruent

Side-Side-Side (SSS) Triangle Congruence Postulate: If three sides in one triangle are congruent to three sides in another triangle, then the triangles are congruent.

Side-Angle-Side (SAS) Triangle Congruence Postulate: If two sides and the included angle in one triangle are congruent to two sides and the included angle in another triangle, then the two triangles are congruent.

Side-Angle-Side (SSS) Triangle Congruence Postulate: If three sides congruent, then the two triangles are congruent.

Similar figures, on the other hand, have the same shape but may differ in size. Shape is intimately related to the notion of proportion, as ancient Egyptian artisans observed long ago. Segments of lengths a, b, c, and d are said to be proportional if a:b = c:d. The fundamental theorem of similarity states that a line segment splits two sides of a triangle into proportional segments if and only if the segment is parallel to the triangle's third side.

The **similarity theorem** may be reformulated as the AAA similarity theorem: two triangles have their corresponding angles equal if and only if their corresponding sides are proportional.

Two similar triangles are related by a scaling (or similarity) factor s: if the first triangle has sides a, b, and c, then the second one will have sides sa, sb, and sc. In addition to the ubiquitous use of scaling factors on construction plans and geographic maps, similarity is fundamental to trigonometry.
An interesting application of these knowledge should be discuss the following problem related to a real a situation.

Problem: The height of the Unknown Soldier: https://mathcitymap.eu/it/portale/#!/task/662767

Calculate the height of the monument to the Unknown Soldier.

The suggestion proposed by the Author aimed to measure the shadow of an object whose height is known and after refer to similar triangles appears evident also looking the following figure used by the same Author of the problem.

1.6. The four important points in a triangle

The **incenter.** Recall that there is a Theorem that show three angle bisectors of a triangle intersect at one point.

Definition. *Thepoint ofintersection of the three anglebisectors of a triangleis called theincenter of the triangle*.

The incenter is denoted by I.

The **centroid**

Definition. In a triangle, the segments thatjoin the vertices with the midpoints of the opposite sides are called medians.

Theorem. In a triangle the three medians intersect at a point, called the centroid of the triangle. The centroid divides each median in the ration2 :1.

The centroid is denoted by G.

The **orthocenter**

Definition. In a triangle ΔABC, the altitude from A is the segment |AD| with D ∈ |BC| that is perpendicular to |BC|.

Note that the altitude from A is unique. A triangle has three altitudes, one for each vertex.

The orthocenter is denoted by H.

The **circumcenter**

Definition. *Given two points in the plane, the perpendicular bisector of the segment |AB|is the locus of the points P in the plane such that |AP|≡|BP|*

It is standard to denote the circumcenter by O.

1.7. Perimeter and Area

In few sentences, just to recall the definition, perimeter for a 2-dimensional shape is the total distance around the respective shape. For the figures with straight sides such as triangle, rectangle, square or a polygon; the perimeter is the sum of lengths for all the sides.

The area for a 2-dimensional shape is the space enclosed within the perimeter of the given shape. To calculate the area for different shapes, use different formulas based on the number of sides and other characteristics such as angles between the sides.

Some students confuse these two concepts, they often show difficulty in understanding the differences between linear (one-dimensional) units and squared (two-dimensional) units or are unable to connect their everyday experience with area and perimeter to what they learn in the classroom.

Area and perimeter instead have to bring together science, math, engineering, city planning, maps, and art, in order to make the perfect STEAM approach in problem solving[2].

The following problem is an example:

Problem 1.2

How large are Perimeter and Area of the penalty area of a regular football field? Is the whole pitch bigger or smaller than a regby pitch?

How many hectares of synthetic grass do you need to cover it entirely?

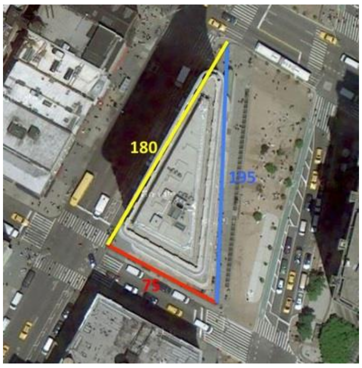
1.8. Pythagoras Theorem

Pythagorean theorem is the well-known geometric theorem that the sum of the squares on the legs of a right triangle is equal to the square on the hypotenuse (the side opposite the right angle)—or, in familiar algebraic notation, $a^2 + b^2 = c^2$.

Although the theorem has long been associated with Greek mathematician-philosopher Pythagoras (c. 570–500/490 BCE), it is actually far older. Four Babylonian tablets from circa 1900–1600 BCE indicate some knowledge of the theorem, with a very accurate calculation of the square root of 2 (the length of the hypotenuse of a right triangle with the length of both legs equal to 1) and lists of special integers known as Pythagorean triples that satisfy it (e.g., 3, 4, and 5). The theorem is mentioned in the Baudhayana Sulba-sutra of India, which was written between 800 and 400 BCE. Nevertheless, the theorem came to be credited to Pythagoras. It is also proposition number 47 from Book I of Euclid's Elements.

Pythagoras theorem can be proved in many ways. Some of the most common and most widely used methods are by using the algebraic method proof and using the similar triangles method to solve them. On the web should be possible to find many teaching resource. For example: https://www.youtube.com/watch?v=r382kfkqAF4

[2] Cfr. https://www.youtube.com/watch?v=cP-MzB6gB80



Possible applications of the Pythagoras Theorem can be traced in Engineering and Construction fields (most architects use the technique of the Pythagorean theorem[3] to find the value as well as when length or breadth are known it is very easy to calculate the diameter of a particular sector), face recognition technics (the distance between the security camera and the place where the person is noted is well projected through the lens using the concept), navigation (people traveling in the sea use this technique to find the shortest distance and route to proceed to their concerned places) etc.

1.9. Circle

A *chord* AB is a segment in the interior of a circle connecting two points (A and B) on the circumference ad in the picture of *Piazza Pretoria* in Palermo. When a chord passes through the circle's centre, it is a diameter, d. The circumference of a circle is given by πd, or $2\pi r$ where r is the *radius* of the circle; the area of a circle is πr^2. In each case, π is the same constant (3.14159...). The Greek mathematician Archimedes (c. 287–212/211 BCE) used the method of exhaustion to obtain upper and lower bounds for π by circumscribing and inscribing regular polygons about a circle.

A semicircle has its end points on a *diameter* of a circle. Thales (6th century BCE) is generally credited with having proved that any angle inscribed in a semicircle is a right angle; that is, for any point C on the semicircle with diameter AB, ∠ ACB will always be 90 degrees (see Sidebar: Thales' Rectangle). Another important theorem states that for any chord AB in a circle, the angle subtended by any point on the same semiarc of the circle will be invariant. Slightly modified, this means that in a circle, equal chords determine equal angles, and vice versa.

1.10. Regular polygons

A *polygon* is called regular if it has equal sides and angles. Thus, a regular triangle is an equilateral triangle, and a regular quadrilateral is a square. A general problem since antiquity has been the problem of constructing a regular n-gon, for different n, with only

[3] Cfr. https://geometryandarchitecture.weebly.com/pythagorean-theorem.html

ruler and compass. Techniques, such as bisecting the angles of known constructions, exist for constructing regular n-gons for many values, but none is known for the general case. In 1797, following centuries without any progress, Gauss surprised the mathematical community by discovering a construction for the 17-gon. More generally, Gauss was able to show that for a prime number p, the regular p-gon is constructible if and only if p is a "Fermat prime": $p = F(k) = 2^{2k} + 1$. Because it is not known in general which F(k) are prime, the construction problem for regular n-gons is still open.

1.11. Rigid Transformations – Isometries

The so-called "elementary geometric transformations" of the plane are *bigective correspondences*, of the plane on itself, characterized by the property of preserving distances (and angles). It is therefore possible to define as a geometric transformation any procedure that allows to obtain from an initial figure F another figure F 'whose points are in one-to-one correspondence with those of F. Figure F will be called transformed or corresponding to figure F in the considered transformation.

Looking to teaching geometry at School we can say that, once this concept has been acquired after numerous examples of transformations carried out by the students also with bricolage materials such as paper, colored cards, transparent sheets, and so on, it will be the teacher's task to stimulate the deepening of the topic, asking the class what has changed or has not changed into a transformed figure with respect to the initial figure.

This will lead the students to the very important concept of transformation invariant, which is that geometric property of the figure (shape, size, position) that does not undergo any alteration during the transformation. The classification of geometric transformations will be based on this concept.

It can therefore be discovered that all geometric transformations, such as congruence, whose invariants are the shape and size of the figures, take the name of isometries (from the Greek *isos*, "equal" and *metron*, "measure").

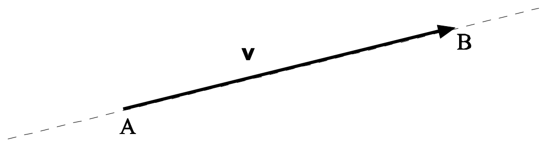
There are only three rigid movements that leave the shape and size of a figure unaltered: *translations, symmetries* and *rotations* either separately or by *combining* them.

Translation

To mathematically characterize this transformation, it is necessary to ask: *how can you* *define the direction of movement, how much does the object or figure move from its initial position, and in which direction?*

Therefore, the need arises to introduce the very important notion of *vector* (from the Latin *vehere,* “to carry”). A vector is an oriented segment, that is, with an arrow at one of the two ends, which indicates the direction in which it must be travelled. The length of the segment is called the module of the vector.

A vector can be indicated in several ways. With reference to the following figure, if A and B are the extremes of the oriented segment, then the vector can be indicated:

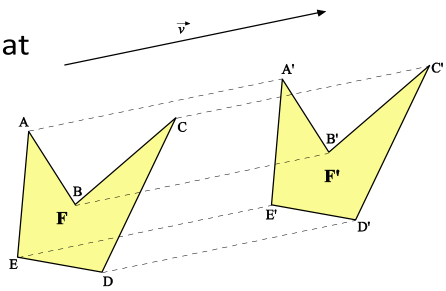


Therefore, a vector is characterized by being a segment that has a direction (that of the line to which the segment belongs), a direction of travel and a module.

The notion of vector will allow us to clarify the concept of translation. In fact, we will say that translation is a rigid movement identified by a vector that establishes the direction, direction and modulus of the movement that occurs in the plane.

The extreme case of translation occurs when the given figure is not translated (ie the so-called null vector) so it represents the identity element of the translations. In this case

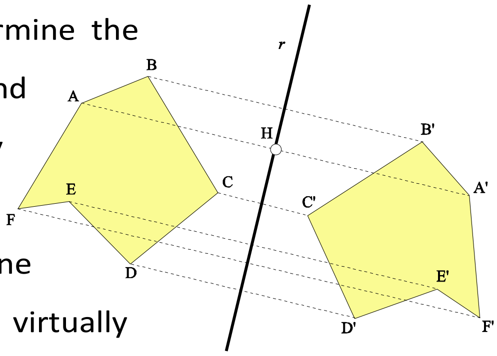
It is said that all the points of the given figure are united, that is, each one corresponds to itself. A figure F and its translated F 'will be said to be directly congruent because the rigid movement occurs on the plane where the starting figure lies.



Symmetries

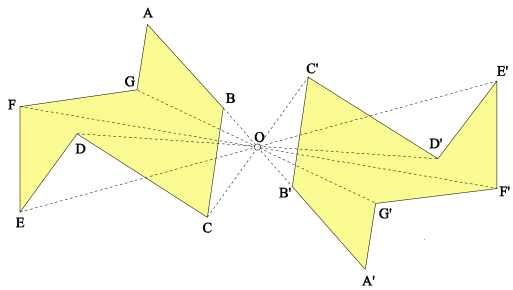
Symmetry is another transformation that, even more than translation, is experienced from the earliest years of life by observing the world around us. In fact, the lack of symmetry in an object, in an animal, in a drawing, in a plant, and so on, immediately catches the eye, even without having specified what kind of symmetry it is.


Knowing how to ascertain, with a glance, axial symmetry means having taken a step forward also as a mathematical competence. To determine the symmetrical image of an object or a geometric figure and therefore to perform an axial symmetry, it is necessary that, given a figure on a plane,
an axis of symmetry is established, that is any straight line of the plane with respect to which the plane it can be virtually folded and superimposed on itself, that is, overturned. What is required is a movement in space like that which occurs when the page of a notebook is turned.



For example, given a figure F and any straight line r of its plane of belonging, which does not intersect F, to obtain the figure F 'symmetric with respect to r, it will be sufficient to trace from each vertex of F, let's set the vertex A, the segment AH perpendicular to r, and on its extension determine with the compass the segment A'H equal to AH. It will be enough to join all the new vertices determined in the aforementioned way to obtain the symmetrical figure F 'of F with respect to the r axis.


The symmetrical arrangement of some architectural elements, some flowers or some small animals, such as protozoa, they will be able to see that in these cases the symmetry cannot be determined with respect to an axis, but rather with respect to a precise point, which is the center of symmetry, with respect to which the symmetrical points of the figure are equidistant.



Since the points are symmetrical about the center of symmetry, this new symmetry comes called central symmetry, and the center will be internal or external to the figure.

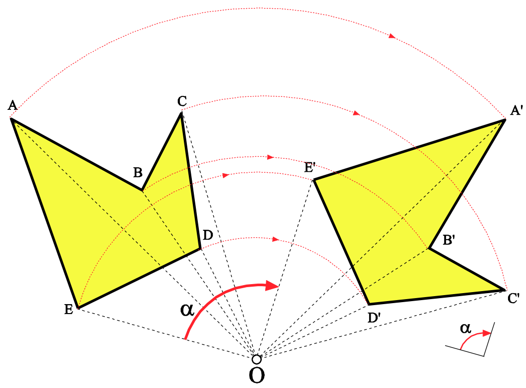
We will say that two figures, F and F ', correspond to each other in a central symmetry when their points are symmetrical with respect to a point of the plane of F and F', called center of symmetry, which turns out to be the midpoint of the segment joining points correspondents.

Rotation

In addition to translation and symmetries, there is another rigid movement that produces congruent figures: rotation.

To rotate the geometric figures, which will be directly congruent, as the movement takes place on the plane that contains them, you can initially work with cut-out cardboard figures, or with transparent sheets.

How can rotation be mathematically characterized? In order for a geometric figure to rotate by a certain angle a on the plane (clockwise or counterclockwise), it is necessary to rotate all its vertices by that angle with respect to a point O of the plane which may be external or internal to the figure, and which it is called the center of rotation. The vertices of the original figure, following the rotation undergone, will assume another position, and joining them will result in the rotated figure.



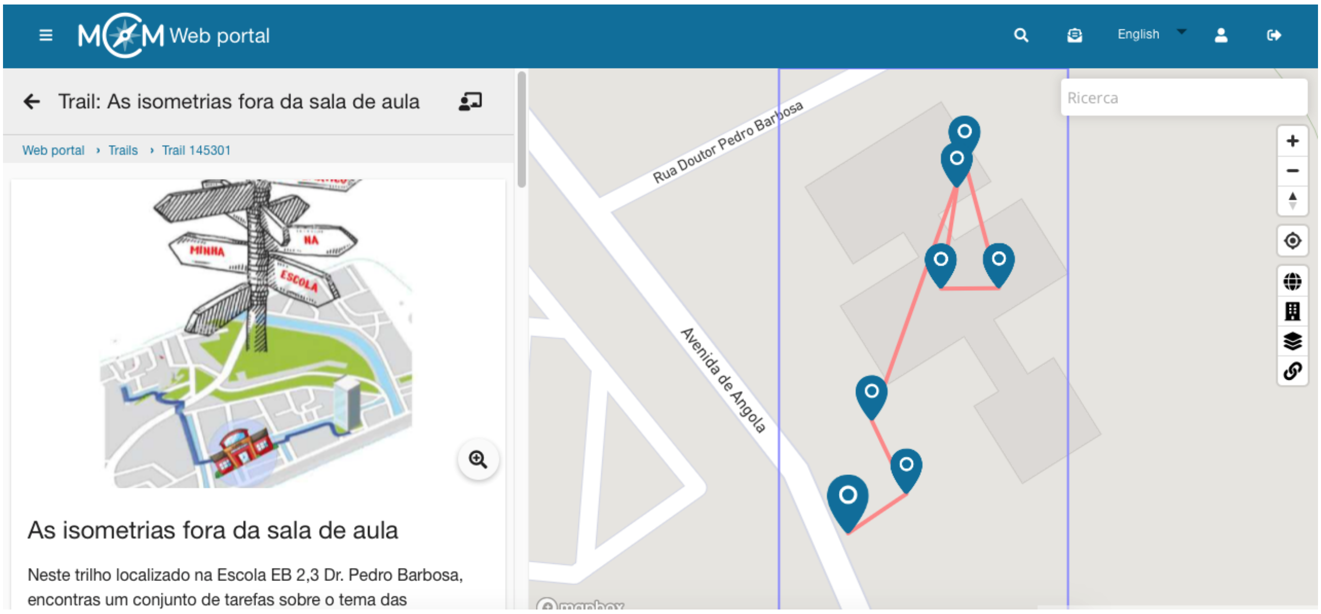
It can therefore be said that rotation is a direct rigid movement identified by a fixed point O, called the center of rotation, and by an angle oriented in a clockwise or counterclockwise direction, which establishes the amplitude of the rotation and the direction in which it it has to happen.

In a rotation, the only joined point is the center of rotation.

Of particular interest is the rotation of a figure by an angle of 180 °. In fact, due to the particular amplitude of the rotation angle, the corresponding points of the two figures will be aligned on opposite half-lines and at the same distance from the center of rotation, so that they are linked by a central symmetry, which can, therefore, be defined as a particular 180 ° rotation.

Interesting application of these knowledge in several real problems should be proposed by the application MapCityMath: https://mathcitymap.eu/en/

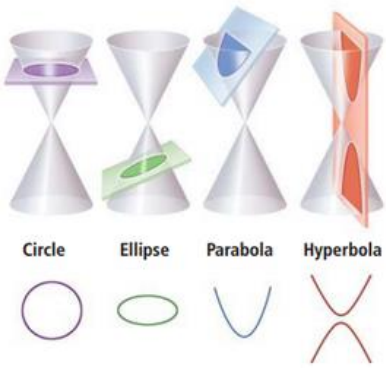
The following figure discusses a significant trail about the use of the isometric transformation to interpret some real contexts trough a mathematical lent.



Vfr: https://mathcitymap.eu/en/portal-en/#!/trail/145301

1.12. Conic sections and geometric art

The most advanced part of plane Euclidean geometry is the theory of the *conic sections* (the ellipse, the parabola, and the hyperbola). Also in this case the application of these construction to our realty is evident[4]



[4] Cfr. https://romaiahm.wordpress.com/2016/08/23/conic-sections-in-architecture/

Much as the Elements displaced all other introductions to geometry, the Conics of Apollonius of Perga c. 240–190 BCE), known by his contemporaries as “the Great Geometer,” was for many centuries the definitive treatise on the subject. Medieval Islamic artists explored ways of using geometric figures for decoration. In the 20th century, internationally renowned artists such as Josef Albers, Max Bill, and Sol LeWitt were inspired by motifs from Euclidean geometry.

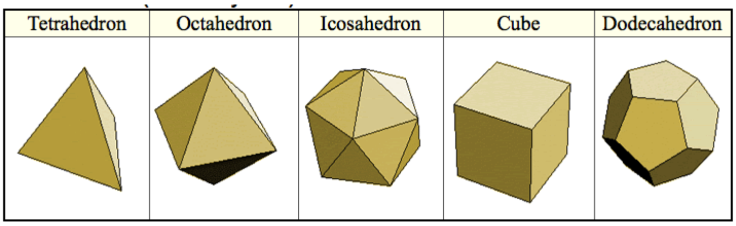
Interesting application of these knowledge in several real problems should be proposed by the application MapCityMath: https://mathcitymap.eu/en/

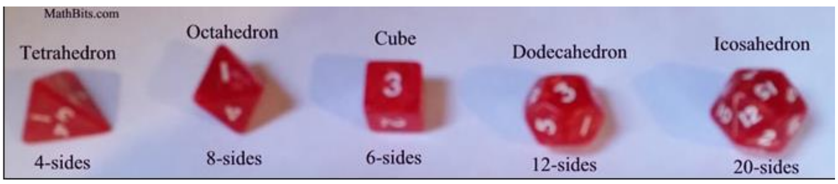
1.13. Solid geometry … the volume

Some concepts, such as proportions and angles, remain unchanged from plane to solid geometry. For other familiar concepts, there exist analogies—most noticeably, volume for area and three-dimensional shapes for two-dimensional shapes (sphere for circle, tetrahedron for triangle, box for rectangle). In plane geometry the area of any polygon can be calculated by dissecting it into triangles. A similar procedure is not possible for solids.

A *Platonic solid* is a regular convex polyhedron in which the faces are congruent regular polygons with the same number of faces meeting at each vertex. (The sum of the internal angles at each vertex is less than 360º.).

There are only five Platonic solids[5]:





Several are the application of solid geometry in real life. The following are two possible

[5] Cfr. http://mathbitsnotebook.com/Geometry/3DShapes/3DPlatonicSolids.html

example:

Problem 1.4

There is a rubbish bin which shaped as perpendicular prism in the picture. What can be the maximum volume of rubbish in the full bin that not hangs over the edge of bin?

Problem 1.5

The figure shows eleven border cylindrical columns. How many kg of red and white painting was necessary for these columns, if 1 kg paint will cover approximately 8 m2 area?

External resources:

With the aim to give to this document also a multimedia setting, giving also some possible external recourses, useful for learning and teaching the same subject we refer the following list:

https://mathcs.clarku.edu/~djoyce/java/elements/elements.html

http://www.itslearning.net/geogebra

https://www.youtube.com/watch?v=aejEwpxfKu8,

https://www.youtube.com/watch?v=r382kfkqAF4 https://www.geogebra.org

https://www.youtube.com/watch?v=cP-MzB6gB80

https://geometryandarchitecture.weebly.com/pythagorean-theorem.html

https://romaiahm.wordpress.com/2016/08/23/conic-sections-in-architecture/

http://mathbitsnotebook.com

https://www.khanacademy.org

Wolfram Problem Generator

https://mathcitymap.eu/en/