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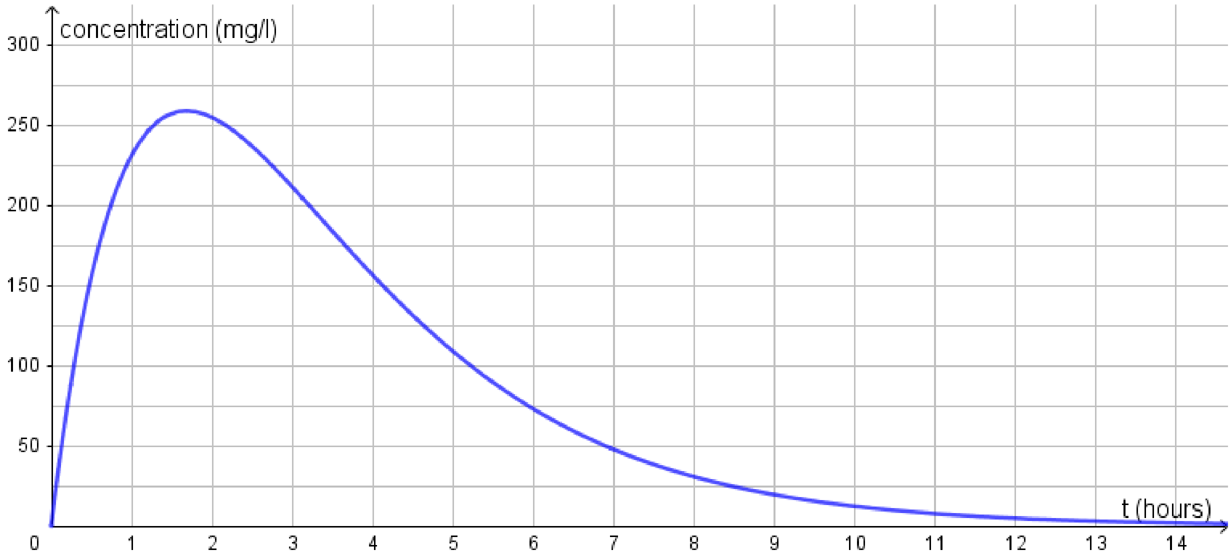
Mathematics Bridging Course
Unit 2e – Introduction to Functions

Introductory Task: A patient is being injected with a certain medicine. The concentration of the active ingredient in the blood of this patient can be described as a function over time. The graph of this function is depicted below.



a) *How high is the concentration after 1 hour?*
b) *At what point in time after the injection does the concentration reach its highest value?*
c) *At what point(s) in time does the concentration reach 150 mg/l?*
d) *At what point in time does the concentration fall most rapidly?*

1. Introduction to functions

A function is basically a way to describe the relations between two sets of mathematical objects, whereby each object of the first set is associated with exactly one object of the second set. In school mathematics, these objects are often real numbers, and one can find the following definition of a *real function* (meaning a function from the real numbers to the real numbers) in several text books:

Definition (draft version): If each element $x \in \mathbb{R}$ is associated with exactly one element $f(x) \in \mathbb{R}$, this relation is called a *real function*, symbolically denoted as $f: \mathbb{R} \to \mathbb{R}$. x is called *argument* or independent variable, $f(x)$ is called *function value* or dependent variable.

This definition works well for a number of functions, e.g. Linear Functions or Polynomials. It is however less suitable for e.g. Rational Functions or Logarithmic Functions, or any other functions that are not defined on the real numbers as a whole, but only on a subset of those. It is therefore more common to amend the above definition to the following one:

Definition: Let A be a subset of the real numbers, i.e. $A \subseteq \mathbb{R}$. If each element $x \in A$ is associated with exactly one element $f(x) \in \mathbb{R}$, this relation is called a *real function*, symbolically expressed as $f: A \to \mathbb{R}$. x is called *argument* or independent variable, $f(x)$ is called *function value* or dependent variable. A is called the *domain* of the function. The second set – i.e. $\mathbb{R}$ – is called the *codomain* of the function.

Example: A sports club has 350 members. Each member is assigned an unique index number a, $a \in \{1, 2, \dots, 350\}$. The function $h: A \to \mathbb{R}$ describes the association between a members' index number and this members' body height.

a) Using this terminology, write a mathematical term that calculates the average body height of all club members!
b) What does the expression $h(s) = h(t)$ mean in this example?

In this example, the domain is a finite set of natural numbers rather than the whole set of real numbers (since it would not make sense to ask for the body height of member number 7.25).

The process of amending a definition also gives us a good example of how mathematics sometimes works. Mathematicians often have an image of certain mathematical objects in their mind and write down a definition of those objects. But by looking at the written-down definition they find that some of the objects that they imagined are not included in this definition, while occasionally other objects that they have not originally imagined are covered by this definition. The definition is then rewritten in a more exact way. Oftentimes, this process repeats several times.

For real functions, the following terminology is also often used in school:

Definition: Let $f: A \to \mathbb{R}$ be a real function. The set I_f of all function values, $I_f = \{f(x) \in \mathbb{R} | x \in A\}$, is called the *image* of f. The set G_f of all pairs of arguments and their corresponding function values, $G_f = \{(x, f(x)) \in \mathbb{R}^2 | x \in A\}$, is called the *graph* of the function.

Since each pair of real numbers can be depicted as a point in a coordinate system, the graph can also be depicted as a set of points in a coordinate system. This depiction is also sometimes called the graph of the function.

In school, the study of functions is usually limited to real functions. But the term can be generalized, as mentioned above, to describe relations between any two sets.

Definition: Let A and B be sets. If each element $x \in A$ is associated with exactly one element $f(x) \in B$, this relation is called a *function*, symbolically expressed as $f: A \to B$. x is called *argument* or independent variable, $f(x)$ is called *function value* or dependent variable. A is called the *domain* of the function, B is called the *codomain* of the function.

At university, special lectures study complex functions (where A and/or B are subsets of the complex numbers) or multidimensional real functions (where A and/or B are subsets of $\mathbb{R}^2, \mathbb{R}^3$, or more generally, $\mathbb{R}^n$). Also, the study of functions is not limited to sets of numbers (or pairs, triplets, … of numbers). One can also study functions between sets of geometric objects (e.g. in Topology) or functions between sets of functions (e.g. in Functional Analysis).